11 – 19 PROGRESSION

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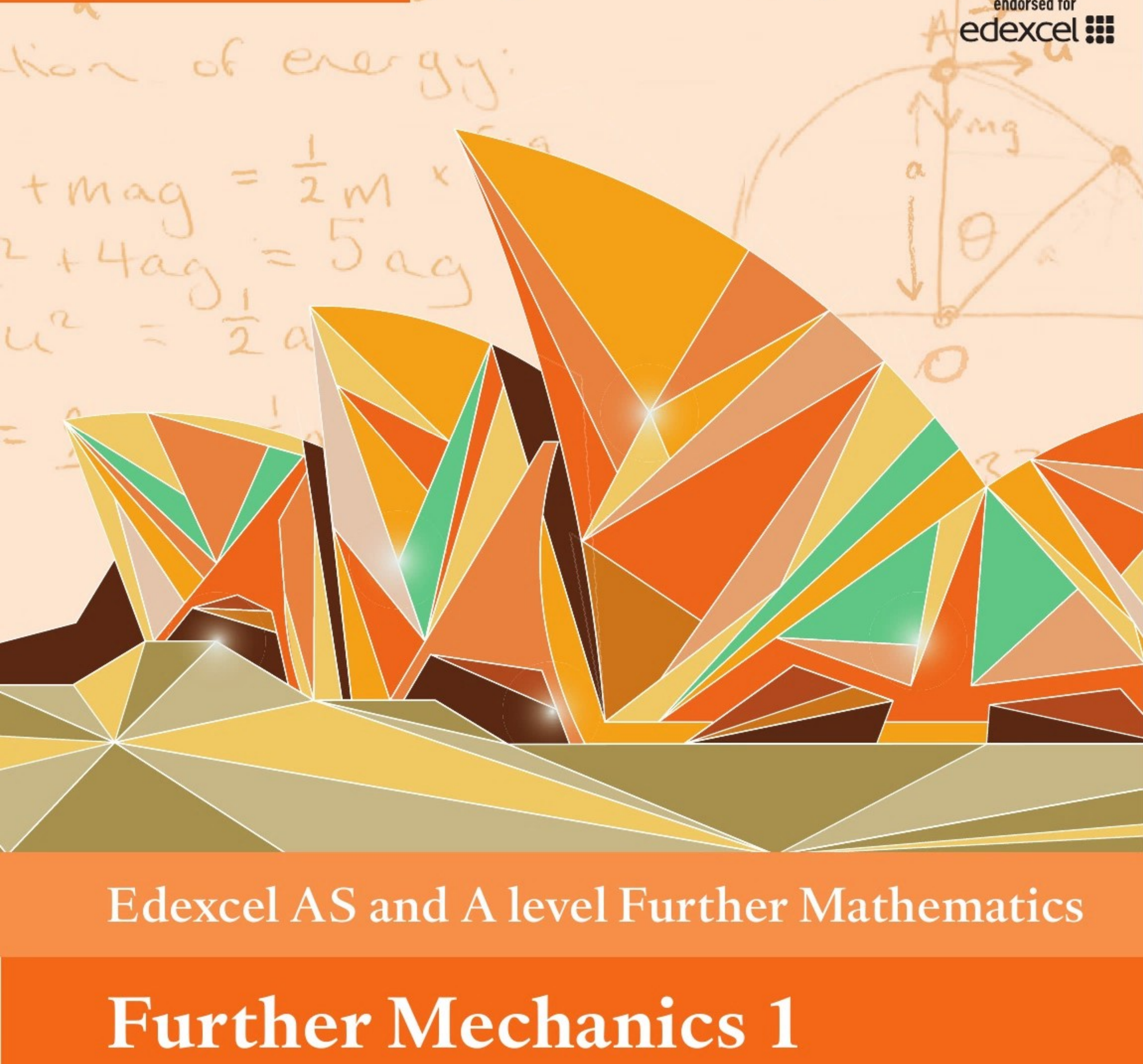

Edexcel AS and A level Further Mathematics

Further Mechanics 1

FM1

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Contents

● = A level only

Overarching themes

The following three overarching themes have been fully integrated throughout the Pearson Edexcel AS and A level Mathematics series, so they can be applied alongside your learning and practice.

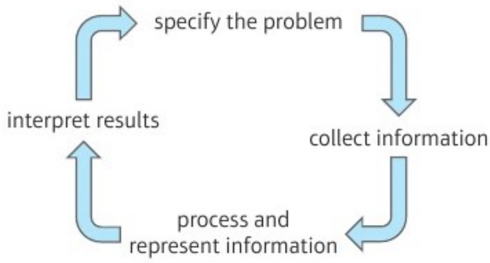
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols
- Dedicated sections on mathematical proof explain key principles and strategies
- Opportunities to critique arguments and justify methods

2. Mathematical problem solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Structured and unstructured questions to build confidence
- Challenge boxes provide extra stretch

The Mathematical Problem-solving cycle



3. Mathematical modelling

- Dedicated modelling sections in relevant topics provide plenty of practice where you need it
- Examples and exercises include qualitative questions that allow you to interpret answers in the context of the model
- Dedicated chapter in Statistics & Mechanics Year 1/AS explains the principles of modelling in mechanics

Finding your way around the book

Access an online digital edition using the code at the front of the book.

Elastic collisions in one dimension

4

Objectives

After completing this chapter you should be able to:

- Solve problems involving the direct impact of two particles by using the principle of conservation of momentum and Newton's law of restitution → pages 70–76
- Apply Newton's law of restitution to problems involving the direct collision of a particle with a smooth plane surface → pages 76–79
- Find the change in energy due to an impact or the application of an impulse → pages 79–84
- Solve problems involving successive direct impacts → pages 84–91

Prior knowledge check

1 Two particles A and B of masses 0.4 kg and 0.5 kg respectively are moving towards each other on a straight line on a smooth horizontal surface. Just before the collision, both particles have speeds of 1 m s⁻¹. After the collision, the direction of motion of B is reversed and its speed is 0.8 m s⁻¹.
a Calculate the speed and direction of A after the collision.
b Calculate the magnitude of the impulse given by A to B during the collision. ← Chapter 1

2 A cricket ball has a mass of 0.16 kg and has kinetic energy of 50 J. Work out the speed of the cricket ball. ← Section 2.1

3 A rock of mass 2 kg falls vertically from the top of a cliff into the sea. Given that the rock is travelling at 26 m s⁻¹ when it hits the water, calculate the height of the cliff. ← Statistics and Mechanics Year 1, Chapter 9

When a ball bounces, the speed with which it leaves the ground cannot be greater than the speed with which it approaches the ground. You can use Newton's law of restitution to model the ratio between these two speeds. → Mixed exercise Q14

65

Each chapter starts with a list of objectives

The real world applications of the maths you are about to learn are highlighted at the start of the chapter with links to relevant questions in the chapter

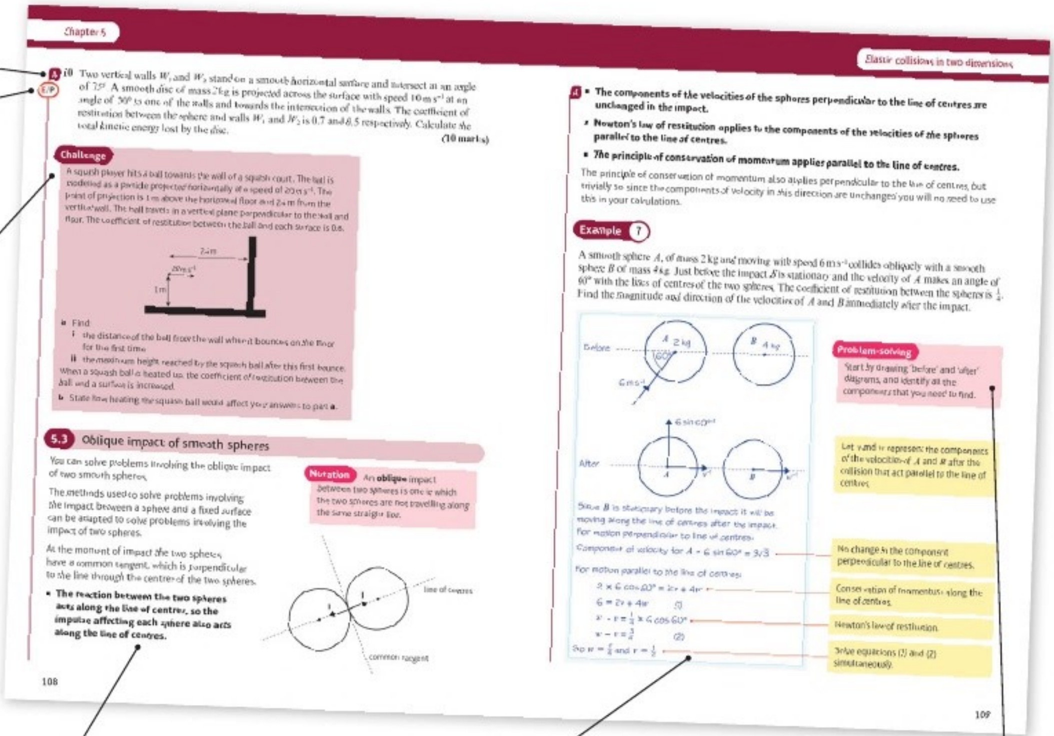
The *Prior knowledge check* helps make sure you are ready to start the chapter

A level content is clearly flagged

Exercises are packed with exam-style questions to ensure you are ready for the exams

Challenge boxes give you a chance to tackle some more difficult questions

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard



Exam-style questions are flagged with (E)

Problem-solving questions are flagged with (P)

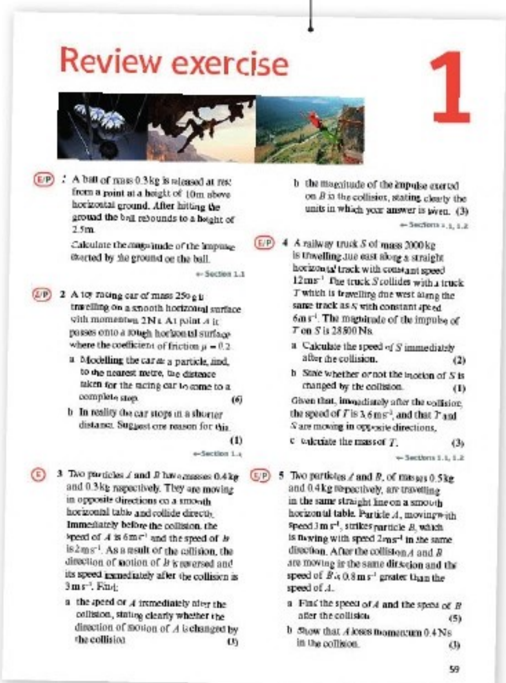
Each section begins with explanation and key learning points

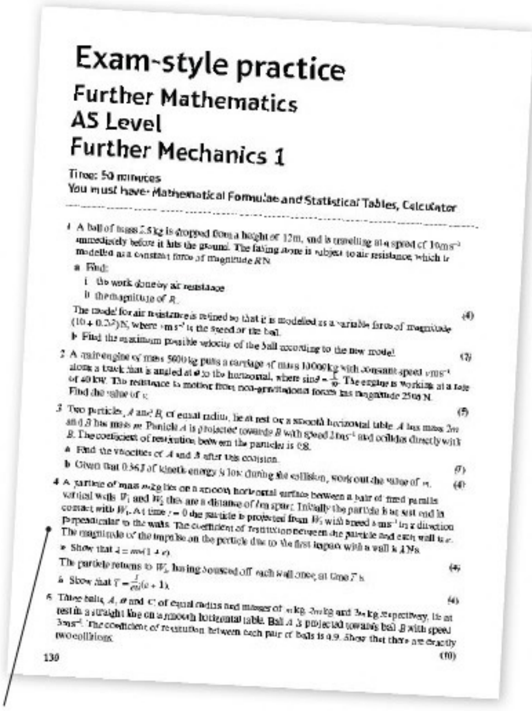
Each chapter ends with a *Mixed exercise* and a *Summary of key points*

Step-by-step worked examples focus on the key types of questions you'll need to tackle

Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

Every few chapters a *Review exercise* helps you consolidate your learning with lots of exam-style questions





AS and A level practice papers at the back of the book help you prepare for the real thing.

Extra online content

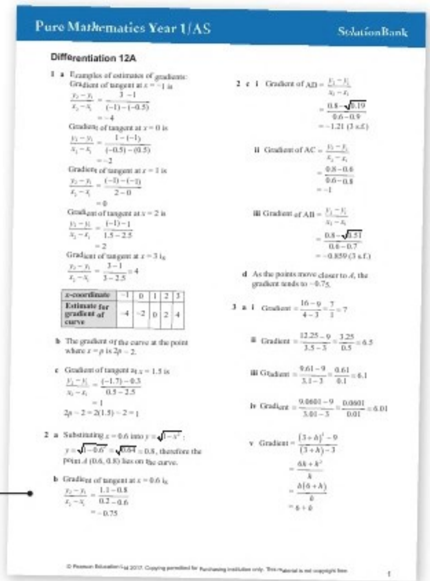
Whenever you see an *Online* box, it means that there is extra online content available to support you.

SolutionBank

SolutionBank provides a full worked solution for every question in the book.

Online Full worked solutions are available in SolutionBank.

Download all the solutions as a PDF or quickly find the solution you need online



Use of technology

Explore topics in more detail, visualise problems and consolidate your understanding using pre-made GeoGebra activities.

Online Find the point of intersection graphically using technology.

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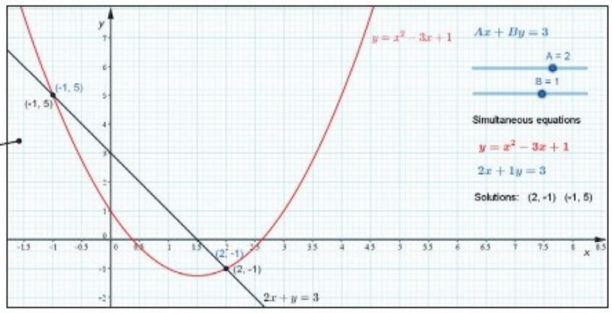

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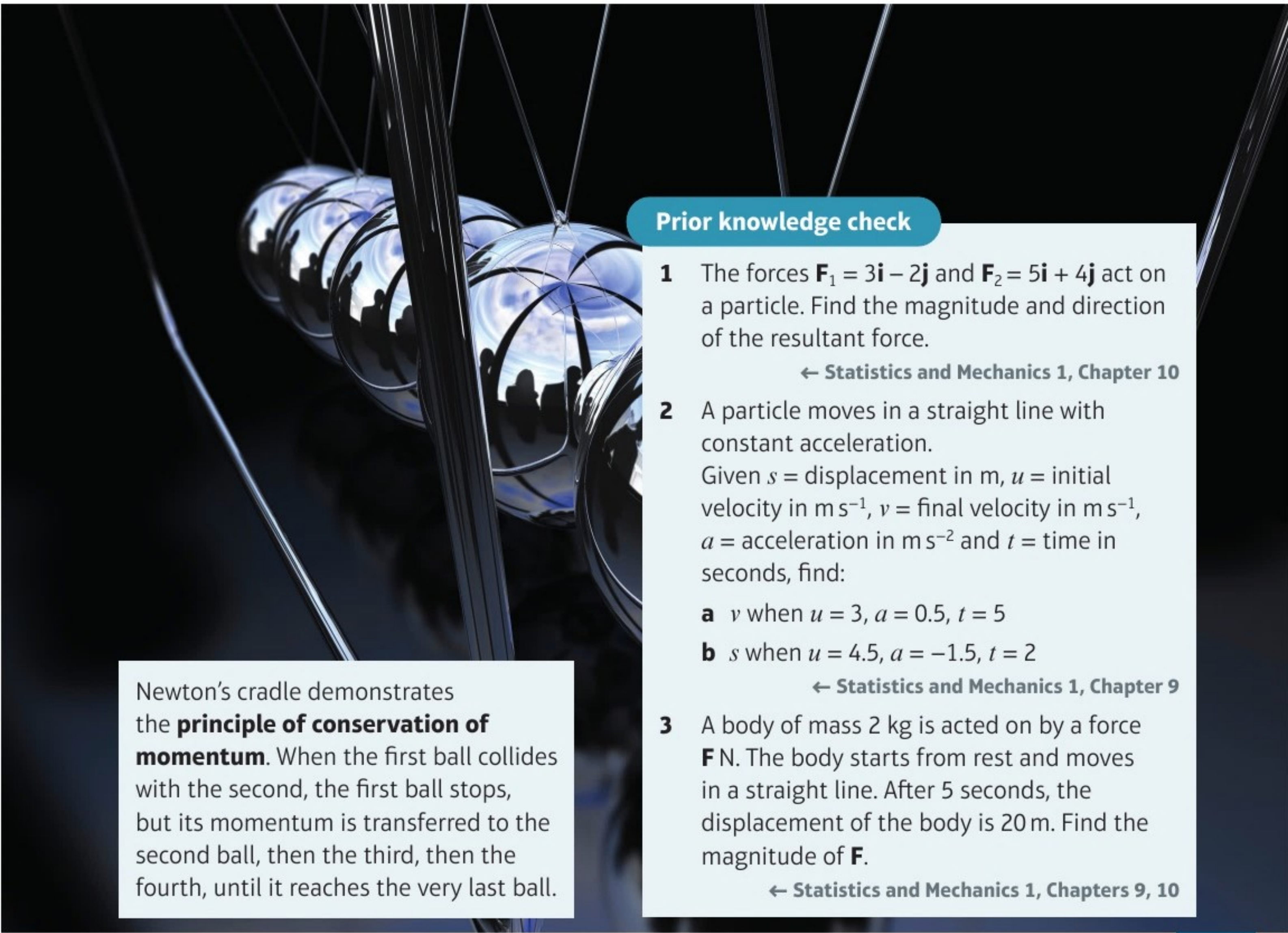
Momentum and impulse

1

Objectives

After completing this chapter you should be able to:

- Calculate the momentum of a particle and the impulse of a force → pages 2–4
- Solve problems involving collisions using the principle of conservation of momentum → pages 4–9
- Use the impulse–momentum principle and the principle of conservation of momentum in vector form → pages 9–12



Newton's cradle demonstrates the **principle of conservation of momentum**. When the first ball collides with the second, the first ball stops, but its momentum is transferred to the second ball, then the third, then the fourth, until it reaches the very last ball.

Prior knowledge check

1 The forces $\mathbf{F}_1 = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{F}_2 = 5\mathbf{i} + 4\mathbf{j}$ act on a particle. Find the magnitude and direction of the resultant force.

← Statistics and Mechanics 1, Chapter 10

2 A particle moves in a straight line with constant acceleration.
Given s = displacement in m, u = initial velocity in m s^{-1}, v = final velocity in m s^{-1}, a = acceleration in m s^{-2} and t = time in seconds, find:

a v when $u = 3$, $a = 0.5$, $t = 5$

b s when $u = 4.5$, $a = -1.5$, $t = 2$

← Statistics and Mechanics 1, Chapter 9

3 A body of mass 2 kg is acted on by a force **F** N. The body starts from rest and moves in a straight line. After 5 seconds, the displacement of the body is 20 m. Find the magnitude of **F**.

← Statistics and Mechanics 1, Chapters 9, 10

1.1 Momentum in one dimension

You can calculate the momentum of a particle and the impulse of a force.

- **The momentum of a body of mass m which is moving with velocity v is mv.**

If m is in kg and v is in m s^{-1} then the units of momentum will be kg m s^{-1}. However, since $\text{kg m s}^{-1} = (\text{kg m s}^{-2})\text{ s}$ and kg m s^{-2} are the units for force ($F = ma$) you can also measure momentum in newton seconds (N s).

Note Velocity is a vector quantity and mass is a scalar, so momentum is a vector quantity.

Example 1

Find the magnitude of the momentum of:

a a cricket ball of mass 400 g moving at 18 m s^{-1}

b a lorry of mass 5 tonnes moving at 0.3 m s^{-1}.

a Momentum = mass × velocity

Magnitude of momentum = $\frac{400}{1000} \times 18$

$= 7.2\text{ N s}$

b Momentum = mass × velocity

Magnitude of momentum = $(5 \times 1000) \times 0.3$

$= 1500\text{ kg m s}^{-1}$

The mass must be in kg.

The units can be N s or kg m s^{-1}.

- **If a constant force F acts for time t then we define the impulse of the force to be Ft.**

If F is in N and t is in s then the units of impulse will be N s.

Note Force is a vector quantity and time is a scalar, so impulse is a vector quantity.

Examples of an impulse include a bat hitting a ball, a snooker ball hitting another ball or a jerk in a string when it suddenly goes tight. In all these cases the time for which the force acts is very small but the force is quite large and so the product of the two, which gives the impulse, is of reasonable size. However, there is no theoretical limit on the size of t.

Suppose a body of mass m is moving with an initial velocity u and is then acted upon by a force F for time t. This results in its final velocity being v.

Its acceleration is given by $a = \frac{v-u}{t}$

Substituting into $F = ma$: $F = m\left(\frac{v-u}{t}\right)$

$$Ft = m(v-u)$$
$$= mv - mu$$

The impulse of the force I is given by $I = Ft$.

- $I = mv - mu$

  Impulse = final momentum − initial momentum

  Impulse = change in momentum

This is called the **impulse–momentum principle**.

Watch out This is a vector equation, so for motion in a straight line a positive direction must be chosen and each value must be given the correct sign.

Example 2

A body of mass 2 kg is initially at rest on a smooth horizontal plane. A horizontal force of magnitude 4.5 N acts on the body for 6 s. Find:

a the magnitude of the impulse given to the body by the force

b the final speed of the body.

a Magnitude of the impulse = force × time
$= 4.5 \times 6$
$= 27$ N s

The units can be N s or kg m s^{-1}.

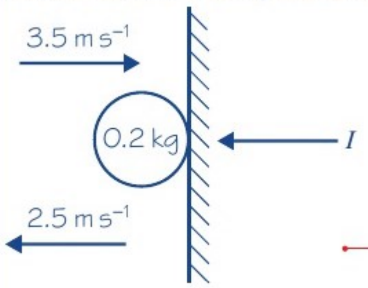
b Impulse = Final momentum − Initial momentum
$27 = 2v - 0$
$v = 13.5$ m s^{-1}

The body is at rest initially.

Example 3

A ball of mass 0.2 kg hits a fixed vertical wall at right angles with speed 3.5 m s^{-1}. The ball rebounds with speed 2.5 m s^{-1}. Find the magnitude of the impulse exerted on the wall by the ball.

This diagram shows the initial and final velocities of the ball and the impulse acting on it.



Problem-solving

Because the wall is fixed you cannot apply the impulse–momentum principle to it. Find the magnitude of the impulse exerted on the ball by the wall and then use Newton's 3rd law to deduce that the magnitude of the impulse exerted on the wall by the ball will be the same.

Note that this is a plan view of the situation.

(←): $I = (0.2 \times 2.5) - (0.2 \times (-3.5))$
$= 0.5 + 0.7$
$= 1.2$ N s

Choose a positive direction (←) and apply the impulse–momentum principle to the ball.

The initial velocity is in the negative direction.

Therefore, by Newton's 3rd law, the magnitude of the impulse exerted on the wall by the ball is 1.2 N s.

Exercise 1A

1 A ball of mass 0.5 kg is at rest when it is struck by a bat and receives an impulse of 15 N s. Find its speed immediately after it is struck.

2 A ball of mass 0.3 kg moving along a horizontal surface hits a fixed vertical wall at right angles with speed 3.5 m s^{-1}. The ball rebounds at right angles to the wall. Given that the magnitude of the impulse exerted on the ball by the wall is 1.8 N s, find the speed of the ball just after it rebounds.

3 A toy car of mass 0.2 kg is pushed from rest along a smooth horizontal floor by a horizontal force of magnitude 0.4 N for 1.5 s. Find its speed at the end of the 1.5 s.

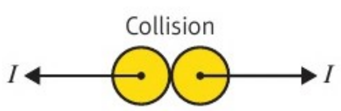
E 4 A ball of mass 0.2 kg, moving along a horizontal surface, hits a fixed vertical wall at right angles. The ball rebounds at right angles to the wall with speed $3.5\,\text{m s}^{-1}$. Given that the magnitude of the impulse exerted on the ball by the wall is 2 N s, find the speed of the ball just before it hits the wall. **(3 marks)**

E/P 5 A ball of mass 0.2 kg is dropped from a height of 2.5 m above horizontal ground. After hitting the ground it bounces to a height of 1.8 m above the ground. Find the magnitude of the impulse received by the ball from the ground. **(4 marks)**

1.2 Conservation of momentum

You can solve problems involving collisions using the principle of conservation of momentum.

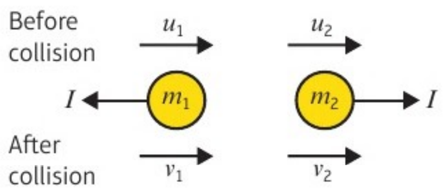
By Newton's 3rd law, when two bodies collide, each one exerts an equal and opposite force on the other. They are in contact for the same time, so they each exert an impulse on the other of equal magnitude but in opposite directions.



By the impulse–momentum principle the changes in momentum of each body are equal but opposite in direction. Thus, these changes in momentum cancel each other out, and the momentum of the whole system is unchanged. This is called the principle of **conservation of momentum**.

- **Total momentum before impact = total momentum after impact**

You can write this in symbols for two masses m_1 and m_2 with velocities u_1 and u_2 respectively before the collision, and velocities v_1 and v_2 respectively after the collision:



- $\boldsymbol{m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2}$

When solving problems involving collisions, always:

- draw a diagram showing the velocities before and after the collision with arrows
- if appropriate, include the impulses on your diagram with arrows
- choose a positive direction and apply the impulse–momentum principle and/or the principle of conservation of momentum.

Example 4

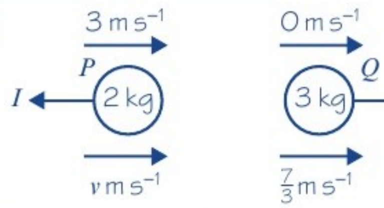
A particle P of mass 2 kg is moving with speed $3\,\text{m s}^{-1}$ on a smooth horizontal plane. Particle Q of mass 3 kg is at rest on the plane. Particle P collides with particle Q and after the collision Q moves off with speed $\frac{7}{3}\,\text{m s}^{-1}$. Find:

a the speed and direction of motion of P after the collision

b the magnitude of the impulse received by P in the collision.

Online Explore particle collisions using GeoGebra.

a



Conservation of momentum: (→)

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

$(2 \times 3) + (3 \times 0) = 2v + \left(3 \times \frac{7}{3}\right)$

$6 = 2v + 7$

$-\frac{1}{2} = v$

The direction of motion of P is reversed by the collision and its speed is $\frac{1}{2}\,\text{m s}^{-1}$.

b For Q: (→) $I = 3\left(\frac{7}{3} - 0\right)$

$= 7\,\text{N s}$

Alternatively, for P: (←)

$I = 2((-v) - (-3))$

$= 2\left(\frac{1}{2} + 3\right)$

$= 7\,\text{N s}$

So the impulse received by P has magnitude $7\,\text{N s}$.

> Draw a diagram showing the velocities before and after the collision (with arrows) and the impulses (with arrows).

> Choose a positive direction and apply the principle of conservation of momentum.

> Since v is negative, P moves in the opposite direction after the collision.

> **Watch out** The direction of motion of P in your answer must be with reference to the original direction of motion of P. Do not use the words left or right.

> To find the impulse consider one particle and apply the impulse–momentum principle. Here it is easier to consider Q.

> Since each particle receives an impulse of equal magnitude, the magnitude of the impulse received by P is also $7\,\text{N s}$.

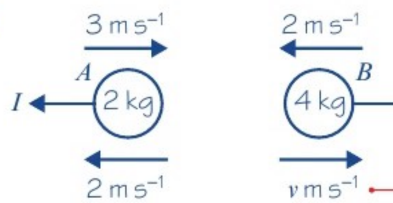
Example 5

Two particles A and B of masses 2 kg and 4 kg respectively are moving towards each other in opposite directions along the same straight line on a smooth horizontal surface. The particles collide. Before the collision the speeds of A and B are $3\,\text{m s}^{-1}$ and $2\,\text{m s}^{-1}$ respectively. After the collision the direction of motion of A is reversed and its speed is $2\,\text{m s}^{-1}$. Find:

a the speed and direction of B after the collision

b the magnitude of the impulse given by A to B in the collision.

a



Conservation of momentum: (→)

$(2 \times 3) + (4 \times (-2)) = (2 \times (-2)) + 4v$

$6 - 8 = -4 + 4v$

$2 = 4v$

$0.5 = v$

B has speed $0.5\,\text{m s}^{-1}$ and its direction of motion is reversed by the collision.

> **Online** Explore collisions with two moving particles using GeoGebra.

> You need to 'guess' the direction of B after the collision. If it is moving in the other direction the answer will be negative.

> This defines the positive direction.

> Each velocity must be given the correct sign.

> As the value of v is positive the 'guess' for the direction of B after the collision was correct.

b For A: (←)

impulse–momentum principle

$I = 2(2 - (-3))$

$= 10\,\text{N s}$

The magnitude of the impulse given by A to B is 10 N s.

Although we could consider either particle, it is safer to consider A since its initial and final speed were given in the question.

The magnitude of the impulse given by A to B is the same as the magnitude of the impulse given by B to A.

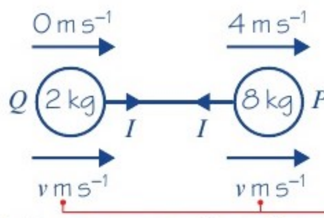
Example 6

Two particles P and Q, of masses 8 kg and 2 kg respectively, are connected by a light inextensible string. The particles are at rest on a smooth horizontal plane with the string slack. Particle P is projected directly away from Q with speed $4\,\text{m s}^{-1}$.

a Find the common speed of the particles after the string goes taut.

b Find the magnitude of the impulse transmitted through the string when it goes taut.

a



The string is inextensible so these are the same.

Using conservation of momentum (→):

This must be applied to the whole system.

$(2 \times 0) + (8 \times 4) = 2v + 8v$

$32 = 10v$

$3.2 = v$

The common speed of the particles is $3.2\,\text{m s}^{-1}$.

b For Q (→):

To find the impulse we must consider one of the particles and apply the impulse–momentum principle. It is easier to consider Q.

$I = 2(v - 0)$

$= 2 \times 3.2$

$= 6.4\,\text{N s}$

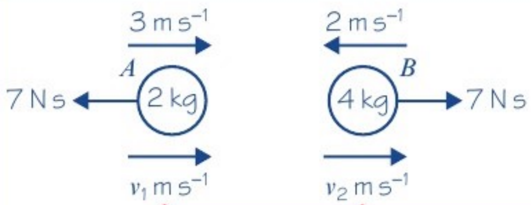
The magnitude of the impulse transmitted through the string (the 'jerk') is 6.4 N s.

Example 7

Two particles A and B of masses 2 kg and 4 kg respectively are moving towards each other in opposite directions along the same straight line on a smooth horizontal surface. The particles collide. Before the collision the speeds of A and B are $3\,\text{m s}^{-1}$ and $2\,\text{m s}^{-1}$ respectively. Given that the magnitude of the impulse due to the collision is 7 N s, find:

a the velocity of A after the collision

b the velocity of B after the collision.



For A: (←)

$7 = 2(v_1 - (-3))$

$7 = 2(v_1 + 3)$

$3.5 = v_1 + 3$

$0.5 = v_1$

For B: (→)

$7 = 4(v_2 - (-2))$

$1.75 = v_2 + 2$

$-0.25 = v_2$

a The direction of motion of A is reversed and its speed is $0.5\,\text{m s}^{-1}$.

b The direction of motion of B is unchanged and its speed is $0.25\,\text{m s}^{-1}$.

Online Explore particle collisions with known impulse using GeoGebra.

The diagram should show the velocities and impulses with arrows.

Again you need to guess which way the particles go after the collision. There are three sensible possibilities: the one shown, $\overleftarrow{v_1}\ \overrightarrow{v_2}$ or $\overleftarrow{v_1}\ \overleftarrow{v_2}$.

To find either v_1 or v_2 consider one particle only, choose a positive direction and apply the impulse–momentum principle.

As v_1 is positive, the guess for the direction of A was correct.

As v_2 is negative, the guess for the direction of B was incorrect and it travels in the opposite direction to that shown on the diagram.

Watch out The question asks for the **velocities** so you need to state the speed and the direction of motion.

Exercise 1B

1 A particle P of mass 2 kg is moving on a smooth horizontal plane with speed $4\,\text{m s}^{-1}$. It collides with a second particle Q of mass 1 kg which is at rest. After the collision P has speed $2\,\text{m s}^{-1}$ and it continues to move in the same direction. Find the speed of Q after the collision.

2 A railway truck of mass 25 tonnes moving at $4\,\text{m s}^{-1}$ collides with a stationary truck of mass 20 tonnes. As a result of the collision the trucks couple together. Find the common speed of the trucks after the collision.

3 Particles A and B have masses 0.5 kg and 0.2 kg respectively. They are moving with speeds $5\,\text{m s}^{-1}$ and $2\,\text{m s}^{-1}$ respectively in the same direction along the same straight line on a smooth horizontal surface when they collide. After the collision A continues to move in the same direction with speed $4\,\text{m s}^{-1}$. Find the speed of B after the collision.

4 A particle of mass 2 kg is moving on a smooth horizontal plane with speed $4\,\text{m s}^{-1}$. It collides with a second particle of mass 1 kg which is at rest. After the collision the particles join together.

a Find the common speed of the particles after the collision.

b Find the magnitude of the impulse in the collision.

(E) **5** Two particles A and B of masses 2 kg and 5 kg respectively are moving towards each other along the same straight line on a smooth horizontal surface. The particles collide. Before the collision the speeds of A and B are $6\,\text{m s}^{-1}$ and $4\,\text{m s}^{-1}$ respectively. After the collision the direction of motion of A is reversed and its speed is $1.5\,\text{m s}^{-1}$. Find:

a the speed and direction of B after the collision **(3 marks)**

b the magnitude of the impulse given by A to B in the collision. **(3 marks)**

E **6** A particle P of mass 150 g is at rest on a smooth horizontal plane. A second particle Q of mass 100 g is projected along the plane with speed $u\,\text{m}\,\text{s}^{-1}$ and collides directly with P. On impact the particles join together and move on with speed $4\,\text{m}\,\text{s}^{-1}$. Find the value of u. **(4 marks)**

E/P **7** A particle A of mass $4m$ is moving along a smooth horizontal surface with speed $2u$. It collides with another particle B of mass $3m$ which is moving with the same speed along the same straight line but in the opposite direction. Given that A is brought to rest by the collision, find:

a the velocity of B after the collision **(3 marks)**

b the magnitude of the impulse given by A to B in the collision. **(3 marks)**

E/P **8** An explosive charge of mass 150 g is designed to split into two parts, one with mass 100 g and the other with mass 50 g. When the charge is moving at $4\,\text{m}\,\text{s}^{-1}$ it splits and the larger part continues to move in the same direction whilst the smaller part moves in the opposite direction. Given that the speed of the larger part is twice the speed of the smaller part, find the speeds of each of the two parts. **(3 marks)**

E/P **9** Two particles P and Q of masses m and km respectively are moving towards each other in opposite directions along the same straight line on a smooth horizontal surface. The particles collide. Before the collision the speeds of P and Q are $3u$ and u respectively. After the collision the direction of motion of both particles is reversed and the speed of each particle is halved.

a Find the value of k. **(4 marks)**

b Find, in terms of m and u, the magnitude of the impulse given by P to Q in the collision. **(3 marks)**

E/P **10** Two particles A and B of masses 4 kg and 2 kg respectively are connected by a light inextensible string. The particles are at rest on a smooth horizontal plane with the string slack. Particle A is projected directly away from B with speed $u\,\text{m}\,\text{s}^{-1}$. When the string goes taut the impulse transmitted through the string has magnitude 6 N s. Find:

a the common speed of the particles just after the string goes taut **(4 marks)**

b the value of u. **(3 marks)**

E/P **11** Two particles P and Q of masses 3 kg and 2 kg respectively are moving along the same straight line on a smooth horizontal surface. The particles collide. After the collision both the particles are moving in the same direction, the speed of P is $1\,\text{m}\,\text{s}^{-1}$ and the speed of Q is $1.5\,\text{m}\,\text{s}^{-1}$. The magnitude of the impulse of P on Q is 9 N s. Find:

a the speed and direction of P before the collision **(3 marks)**

b the speed and direction of Q before the collision. **(3 marks)**

E/P **12** Two particles A and B are moving in the same direction along the same straight line on a smooth horizontal surface. The particles collide. Before the collision the speed of B is $1.5\,\text{m}\,\text{s}^{-1}$. After the collision the direction of motion of both particles is unchanged, the speed of A is $2.5\,\text{m}\,\text{s}^{-1}$ and the speed of B is $3\,\text{m}\,\text{s}^{-1}$. Given that the mass of A is three times the mass of B,

a find the speed of A before the collision. **(4 marks)**

Given that the magnitude of the impulse on A in the collision is 3 N s,

b find the mass of A. **(3 marks)**

Challenge

Particle P has mass $3m$ kg and particle Q has mass m kg. The particles are moving in opposite directions along the same straight line on a smooth horizontal plane when they collide directly. Immediately before the collision, the speed of P is u_1 m s^{-1} and the speed of Q is u_2 m s^{-1}. In the collision, the direction of motion of P is unchanged and the direction of Q is reversed. Immediately after the collision, the speed of P is $\frac{1}{4}u_1$ and the speed of Q is $\frac{1}{2}u_2$. Show that $u_1 = \frac{2}{3}u_2$.

1.3 Momentum as a vector

A

You have used the impulse–momentum principle and the principle of conservation of linear momentum for motion in one dimension.

The **impulse–momentum principle** states that the impulse of a force is equal to the change in momentum:

impulse = force × time

$$I = mv - mu$$

Impulse is measured in newton seconds (N s).

For two-particle collisions, the **principle of conservation of momentum** states that the total momentum before impact equals the total momentum after impact:

momentum = mass × velocity

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Momentum is measured in newton seconds (N s) or kg m s^{-1}.

Impulse and momentum are both vector quantities. You can write the impulse–momentum principle and the principle of conservation of momentum as vector equations, and use them to solve problems involving collisions where the velocities and any impulse are given in vector form.

- $\mathbf{I} = m\mathbf{v} - m\mathbf{u}$
  where m is the mass of the body, u the initial velocity and v the final velocity.
- $m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$
  where a body of mass m_1 moving with velocity $\mathbf{u}_1$ collides with a body of mass m_2 moving with a velocity of $\mathbf{u}_2$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of the bodies after the collision.

Example 8

A particle of mass 0.2 kg is moving with velocity $(10\mathbf{i} - 5\mathbf{j})$ m s^{-1} when it receives an impulse $(3\mathbf{i} - 2\mathbf{j})$ N s. Find the new velocity of the particle.

The change in momentum of the particle is

$0.2\mathbf{v} - 0.2(10\mathbf{i} - 5\mathbf{j})$ N s

Let the velocity of the particle after the impact be $\mathbf{v}$ m s^{-1}.

From the impulse–momentum principle this is equal to the impulse:

$0.2\mathbf{v} - 0.2(10\mathbf{i} - 5\mathbf{j}) = 3\mathbf{i} - 2\mathbf{j}$

Use $m\mathbf{v} - m\mathbf{u} = \mathbf{I}$ substituting $m = 0.2$, $\mathbf{u} = (10\mathbf{i} - 5\mathbf{j})$ and $\mathbf{I} = 3\mathbf{i} - 2\mathbf{j}$

$0.2\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{i} - \mathbf{j}$

$= 5\mathbf{i} - 3\mathbf{j}$

$\mathbf{v} = 25\mathbf{i} - 15\mathbf{j}$

Make $\mathbf{v}$ the subject.

Example 9

A

An ice hockey puck of mass 0.17 kg receives an impulse **Q** N s. Immediately before the impulse the velocity of the puck is $(10\mathbf{i} + 5\mathbf{j})$ m s^{-1} and immediately afterwards its velocity is $(15\mathbf{i} - 7\mathbf{j})$ m s^{-1}. Find the magnitude of **Q** and the angle between **Q** and **i**.

Impulse = change in momentum

$\mathbf{Q} = m\mathbf{v} - m\mathbf{u}$

$\mathbf{Q} = 0.17(15\mathbf{i} - 7\mathbf{j}) - 0.17(10\mathbf{i} + 5\mathbf{j})$

$= 0.17(15\mathbf{i} - 7\mathbf{j} - 10\mathbf{i} - 5\mathbf{j})$

$= 0.17(5\mathbf{i} - 12\mathbf{j})$

$= 0.85\mathbf{i} - 2.04\mathbf{j}$

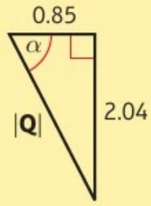
$|\mathbf{Q}| = \sqrt{0.85^2 + (-2.04)^2}$

$= \sqrt{4.8841}$

$= 2.21$

The angle α between **Q** and **i** is $\arctan\left(\frac{2.04}{0.85}\right)$ which is 67.4° (1 d.p.).

Substitute $m = 0.17$, $\mathbf{u} = 10\mathbf{i} + 5\mathbf{j}$ and $\mathbf{v} = 15\mathbf{i} - 7\mathbf{j}$ into the impulse–momentum equation.



Use Pythagoras' theorem to find the magnitude |**Q**| and trigonometry to find the angle.

Example 10

A squash ball of mass 0.025 kg is moving with velocity $(22\mathbf{i} + 37\mathbf{j})$ m s^{-1} when it hits a wall. It rebounds with velocity $(10\mathbf{i} - 11\mathbf{j})$ m s^{-1}. Find the impulse exerted by the wall on the squash ball.

Impulse $= m\mathbf{v} - m\mathbf{u}$

Impulse $= 0.025((10\mathbf{i} - 11\mathbf{j}) - (22\mathbf{i} + 37\mathbf{j}))$

$= 0.025(-12\mathbf{i} - 48\mathbf{j})$

$= (-0.3\mathbf{i} - 1.2\mathbf{j})$ N s

The impulse exerted by the wall on the squash ball is equal to the change in momentum of the ball.

Example 11

A particle of mass 0.15 kg is moving with velocity $(20\mathbf{i} - 10\mathbf{j})$ m s^{-1} when it collides with a particle of mass 0.25 kg moving with velocity $(16\mathbf{i} - 8\mathbf{j})$ m s^{-1}. The two particles coalesce and form one particle of mass 0.4 kg. Find the velocity of the combined particle.

$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$

$0.15(20\mathbf{i} - 10\mathbf{j}) + 0.25(16\mathbf{i} - 8\mathbf{j}) = 0.4\mathbf{v}$

$3\mathbf{i} - 1.5\mathbf{j} + 4\mathbf{i} - 2\mathbf{j} = 0.4\mathbf{v}$

$7\mathbf{i} - 3.5\mathbf{j} = 0.4\mathbf{v}$

$\mathbf{v} = 17.5\mathbf{i} - 8.75\mathbf{j}$

The velocity of the combined particle is $(17.5\mathbf{i} - 8.75\mathbf{j})$ m s^{-1}.

Online Explore particle collisions in two dimensions using GeoGebra.

This is the vector form of the conservation of momentum equation.

After the impact $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}$ and the equation becomes $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{v}$.

Note that the velocity vectors are all parallel and the question involves direct impact. Oblique impact is covered later. → Chapter 5

Exercise 1C

A

In this exercise **i** and **j** are perpendicular unit vectors.

1 A particle of mass 0.25 kg is moving with velocity $(12\mathbf{i} + 4\mathbf{j})\,\text{m s}^{-1}$ when it receives an impulse $(8\mathbf{i} - 7\mathbf{j})\,\text{N s}$. Find the new velocity of the particle.

2 A particle of mass 0.5 kg is moving with velocity $(2\mathbf{i} - 2\mathbf{j})\,\text{m s}^{-1}$ when it receives an impulse $(3\mathbf{i} + 5\mathbf{j})\,\text{N s}$. Find the new velocity of the particle.

3 A particle of mass 2 kg moves with velocity $(3\mathbf{i} + 2\mathbf{j})\,\text{m s}^{-1}$ immediately after it has received an impulse $(4\mathbf{i} + 8\mathbf{j})\,\text{N s}$. Find the original velocity of the particle.

4 A particle of mass 1.5 kg moves with velocity $(5\mathbf{i} - 8\mathbf{j})\,\text{m s}^{-1}$ immediately after it has received an impulse $(3\mathbf{i} - 6\mathbf{j})\,\text{N s}$. Find the original velocity of the particle.

5 A body of mass 3 kg is initially moving with a constant velocity of $(\mathbf{i} + \mathbf{j})\,\text{m s}^{-1}$ when it is acted on by a force of $(6\mathbf{i} - 8\mathbf{j})\,\text{N}$ for 3 seconds. Find the impulse exerted on the body and find its velocity when the force ceases to act.

6 A body of mass 0.5 kg is initially moving with a constant velocity of $(5\mathbf{i} + 12\mathbf{j})\,\text{m s}^{-1}$ when it is acted on by a force of $(2\mathbf{i} - \mathbf{j})\,\text{N}$ for 5 seconds. Find the impulse exerted on the body and find its velocity when the force ceases to act.

7 A particle of mass 2 kg is moving with velocity $(5\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$ when it hits a wall. It rebounds with velocity $(-\mathbf{i} - 3\mathbf{j})\,\text{m s}^{-1}$. Find the impulse exerted by the wall on the particle.

8 A particle of mass 0.5 kg is moving with velocity $(11\mathbf{i} - 2\mathbf{j})\,\text{m s}^{-1}$ when it hits a wall. It rebounds with velocity $(-\mathbf{i} + 7\mathbf{j})\,\text{m s}^{-1}$. Find the impulse exerted by the wall on the particle.

9 A particle P of mass 3 kg receives an impulse $\mathbf{Q}$ Ns. Immediately before the impulse the velocity of P is $5\mathbf{i}\,\text{m s}^{-1}$ and immediately afterwards it is $(13\mathbf{i} - 6\mathbf{j})\,\text{m s}^{-1}$. Find the magnitude of **Q** and the angle between **Q** and **i**.

10 A particle P of mass 0.5 kg receives an impulse $\mathbf{Q}$ Ns. Immediately before the impulse the velocity of P is $(-\mathbf{i} - 2\mathbf{j})\,\text{m s}^{-1}$ and immediately afterwards it is $(3\mathbf{i} - 4\mathbf{j})\,\text{m s}^{-1}$. Find the magnitude of **Q** and the angle between **Q** and **i**.

E 11 A cricket ball of mass 0.5 kg is hit by a bat. Immediately before being hit the velocity of the ball is $(20\mathbf{i} - 4\mathbf{j})\,\text{m s}^{-1}$ and immediately afterwards it is $(-16\mathbf{i} + 8\mathbf{j})\,\text{m s}^{-1}$. Find the magnitude of the impulse exerted on the ball by the bat. **(3 marks)**

E 12 A ball of mass 0.2 kg is hit by a bat. Immediately before being hit by the bat the velocity of the ball is $-15\mathbf{i}\,\text{m s}^{-1}$ and the bat exerts an impulse of $(2\mathbf{i} + 6\mathbf{j})\,\text{N s}$ on the ball. Find the velocity of the ball after the impact. **(3 marks)**

E 13 A particle of mass 0.25 kg has velocity $\mathbf{v}\,\text{m s}^{-1}$ at time t s where $\mathbf{v} = (t^2 - 3)\mathbf{i} + 4t\mathbf{j}$, $t \leqslant 3$. When $t = 3$, the particle receives an impulse of $(2\mathbf{i} + 2\mathbf{j})\,\text{N s}$. Find the velocity of the particle immediately after the impulse. **(3 marks)**

E/P 14 A ball of mass 2 kg is initially moving with a velocity of $(\mathbf{i} + \mathbf{j})\,\text{m s}^{-1}$. It receives an impulse of $2\mathbf{j}$ Ns. Find the velocity immediately after the impulse and the angle through which the ball is deflected as a result. Give your answer to the nearest degree. **(5 marks)**

A E **15** A particle of mass 0.5 kg moving with velocity $3\mathbf{i}$ m s^{-1} collides with a particle of mass 0.25 kg moving with velocity $12\mathbf{i}$ m s^{-1}. The two particles coalesce and move as one particle of mass 0.75 kg. Find the velocity of the combined particle. **(3 marks)**

E **16** A particle of mass 5 kg moving with velocity $(\mathbf{i} - \mathbf{j})$ m s^{-1} collides with a particle of mass 2 kg moving with velocity $(-\mathbf{i} + \mathbf{j})$ m s^{-1}. The two particles coalesce and move as one particle of mass 7 kg. Find the magnitude of the velocity $\mathbf{v}$ m s^{-1} of the combined particle. **(3 marks)**

Challenge

A particle of mass m kg moves with a constant velocity of $(a\mathbf{i} + b\mathbf{j})$ m s^{-1}. After being given an impulse, the particle then moves with a constant velocity of $(c\mathbf{i} + d\mathbf{j})$ m s^{-1}. Given that the direction of the impulse makes an angle of 45° above the direction of $\mathbf{i}$, show that $b + c = a + d$.

Mixed exercise 1

P **1** A particle P of mass $3m$ is moving along a straight line with constant speed $2u$. It collides with another particle Q of mass $4m$ which is moving with speed u along the same line but in the opposite direction. As a result of the collision P is brought to rest.

a Find the speed of Q after the collision and state its direction of motion.

b Find the magnitude of the impulse exerted by Q on P in the collision.

E/P **2** A pile driver of mass 1000 kg drives a pile of mass 200 kg vertically into the ground. The driver falls freely a vertical distance of 10 m before hitting the pile. Immediately after the driver impacts with the pile it can be assumed that they both move with the same velocity. By modelling the pile and the driver as particles, find:

a the speed of the driver immediately before it hits the pile **(2 marks)**

b the common speed of the pile and driver immediately after the impact. **(3 marks)**

The ground provides a constant resistance to the motion of the pile driver of magnitude 120 000 N.

c Find the distance that the pile is driven into the ground before coming to rest. **(2 marks)**

d Comment on this model in relation to the motion of the pile and driver immediately after impact. **(1 mark)**

E **3** A car of mass 800 kg is travelling along a straight horizontal road. A constant retarding force of F N reduces the speed of the car from 18 m s^{-1} to 12 m s^{-1} in 2.4 s. Calculate:

a the value of F **(4 marks)**

b the distance moved by the car in these 2.4 s. **(3 marks)**

E **4** Two particles A and B, of masses 0.2 kg and 0.3 kg respectively, are free to move in a smooth horizontal groove. Initially B is at rest and A is moving toward B with a speed of 4 m s^{-1}. After the impact the speed of B is 1.5 m s^{-1}. Find:

a the speed of A after the impact **(3 marks)**

b the magnitude of the impulse of B on A during the impact. **(3 marks)**

E **5** A railway truck P of mass 2000 kg is moving along a straight horizontal track with speed 10 m s^{-1}. The truck P collides with a truck Q of mass 3000 kg, which is at rest on the same track. Immediately after the collision Q moves with speed 5 m s^{-1}. Calculate:

a the speed of P immediately after the collision **(3 marks)**

b the magnitude of the impulse exerted by P on Q during the collision. **(3 marks)**

E **6** A particle P of mass 1.5 kg is moving along a straight horizontal line with speed $3\,\text{m s}^{-1}$. Another particle Q of mass 2.5 kg is moving, in the opposite direction, along the same straight line with speed $4\,\text{m s}^{-1}$. The particles collide. Immediately after the collision the direction of motion of P is reversed and its speed is $2.5\,\text{m s}^{-1}$.

a Calculate the speed of Q immediately after the impact. **(3 marks)**

b State whether or not the direction of motion of Q is changed by the collision. **(1 mark)**

c Calculate the magnitude of the impulse exerted by Q on P, giving the units of your answer. **(3 marks)**

E/P **7** A particle A of mass m is moving with speed $2u$ in a straight line on a smooth horizontal table. It collides with another particle B of mass km which is moving in the same straight line on the table with speed u in the opposite direction to A. In the collision, the particles form a single particle which moves with speed $\frac{2}{3}u$ in the original direction of A's motion.

Find the value of k. **(3 marks)**

E/P **8** A metal pin of mass 2 kg is driven vertically into the ground by a blow from a sledgehammer of mass 10 kg. The hammer falls vertically on to the pin, its speed just before impact being $9\,\text{m s}^{-1}$. In a model of the situation it is assumed that, after impact, the pin and the hammer stay in contact and move together before coming to rest.

a Find the speed of the pin immediately after impact. **(3 marks)**

The pin moves 3 cm into the ground before coming to rest. Assuming in this model that the ground exerts a constant resistive force of magnitude R newtons as the pin is driven down,

b find the value of R. **(5 marks)**

c State one way in which this model might be refined to be more realistic. **(1 mark)**

A E **9** A cricket ball of mass 0.5 kg is struck by a bat. Immediately before being struck the velocity of the ball is $-25\mathbf{i}\,\text{m s}^{-1}$. Immediately after being struck the velocity of the ball is $(23\mathbf{i} + 20\mathbf{j})\,\text{m s}^{-1}$. Find the magnitude of the impulse exerted on the ball by the bat and the angle between the impulse and the direction of $\mathbf{i}$. **(5 marks)**

E **10** A ball of mass 0.2 kg is hit by a bat which gives it an impulse of $(2.4\mathbf{i} + 3.6\mathbf{j})\,\text{N s}$. The velocity of the ball immediately after being hit is $(12\mathbf{i} + 5\mathbf{j})\,\text{m s}^{-1}$. Find the velocity of the ball immediately before it is hit. **(3 marks)**

E **11** A body P of mass 4 kg is moving with velocity $(2\mathbf{i} + 16\mathbf{j})\,\text{m s}^{-1}$ when it collides with a body Q of mass 3 kg moving with velocity $(-\mathbf{i} - 8\mathbf{j})\,\text{m s}^{-1}$. Immediately after the collision the velocity of P is $(-4\mathbf{i} - 32\mathbf{j})\,\text{m s}^{-1}$. Find the velocity of Q immediately after the collision. **(3 marks)**

E/P **12** A particle P of mass 0.3 kg is moving so that its position vector $\mathbf{r}$ metres at time t seconds is given by

$$\mathbf{r} = (t^3 + t^2 + 4t)\mathbf{i} + (11t)\mathbf{j},\ t \leqslant 4.$$

a Calculate the speed of P when $t = 4$. **(4 marks)**

When $t = 4$, the particle is given an impulse $(2.4\mathbf{i} + 3.6\mathbf{j})\,\text{N s}$.

b Find the velocity of P immediately after the impulse. **(3 marks)**

Challenge

A particle P of mass m kg moves at a speed of $u\,\text{m}\,\text{s}^{-1}$ when it collides head on with a particle Q of mass km kg also travelling at a speed of $u\,\text{m}\,\text{s}^{-1}$, travelling along the same line.

After collision, the two particles move off together with a common speed of $v\,\text{m}\,\text{s}^{-1}$.

Show that k can be written as $\frac{u-v}{u+v}$ or $\frac{u+v}{u-v}$ and explain how these expressions relate to:

a the relative size of u and v

b the subsequent motion of P and Q after collision.

Summary of key points

1 The **momentum** of a body of mass m which is moving with velocity v is mv. The units of momentum can be N s or kg m s^{-1}.

2 If a constant force F acts for a time t then we define the **impulse** of the force to be Ft. The units of impulse are N s.

3 The **impulse–momentum principle**:

Impulse = final momentum – initial momentum

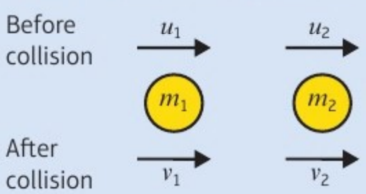
Impulse = change in momentum

$$I = mv - mu$$

where m is the mass of the body, u the initial velocity and v the final velocity.

4 **Principle of conservation of momentum:**

Total momentum before impact = total momentum after impact



$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

where a body of mass m_1 moving with velocity u_1 collides with a body of mass m_2 moving with velocity u_2, v_1 and v_2 are the velocities of m_1 and m_2 after the collision respectively.

A

5 You can write the impulse–momentum principle and the principle of conservation of momentum as vector equations

- $\mathbf{I} = m\mathbf{v} - m\mathbf{u}$
  where m is the mass of the body, $\mathbf{u}$ the initial velocity and $\mathbf{v}$ the final velocity.
- $m_1\mathbf{u_1} + m_2\mathbf{u_2} = m_1\mathbf{v_1} + m_2\mathbf{v_2}$
  where a body of mass m_1 moving with velocity $\mathbf{u_1}$ collides with a body of mass m_2 moving with velocity $\mathbf{u_2}$, $\mathbf{v_1}$ and $\mathbf{v_2}$ are the velocities of the bodies after the collision.

Work, energy and power

2

Objectives

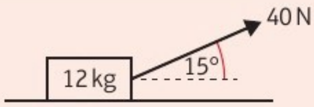
After completing this chapter you should be able to:

- Calculate the work done by a force when its point of application moves → pages 16–19
- Calculate the kinetic energy of a moving particle and the potential energy of a particle → pages 20–23
- Use the principle of conservation of mechanical energy and the work–energy principle → pages 24–28
- Calculate the power developed by an engine → pages 29–33

This rock climber is increasing her height above sea-level. Her gravitational potential energy is increasing. When she abseils back down the rock face, her gravitational potential energy will be converted into kinetic energy. → Exercise 2B Q11

Prior knowledge check

1



A crate of mass 12 kg is at rest on a smooth horizontal plane. It is dragged by means of a force of magnitude 40 N, which acts at an angle of 15° above the horizontal. Find:

a the magnitude of the normal reaction of the plane on the box

b the acceleration of the box

c the total distance travelled by the box in the first 5 seconds of its motion.

← Statistics and Mechanics Year 2, Chapter 5

2 A 10 kg box rests on a rough plane inclined at 30° to the horizontal. Given that the box is on the point of slipping down the plane, find the coefficient of friction between the box and the plane.

← Statistics and Mechanics Year 2, Chapter 5

2.1 Work done

If you drag an object along the ground, you have to apply a force to overcome friction. In order to move the object you have to do **work**. In general, work is done on an object when a force is applied to it and there is motion.

- **You can calculate the work done by a force when its point of application moves along a straight line using the formula:**

work done = component of force in direction of motion × distance moved in direction of force

When the force is measured in newtons and the distance moved in metres, the work done is measured in joules (J).

Note If the point of application is moving in the same direction as the line of action of the force, this formula becomes: **work done = force × distance**

You can also calculate the work done against gravity when a particle is moved vertically. Work is done against gravity whenever a particle's vertical height is increased. This may be because the particle moved vertically or at an angle to the horizontal.

- **Work done against gravity = mgh, where m is the mass of the particle, g is the acceleration due to gravity and h is the vertical distance raised.**

Example 1

A box is pulled 7 m across a horizontal floor by a horizontal force of magnitude 15 N.
Calculate the work done by the force.

Work $= Fs$

$= 15 \times 7$

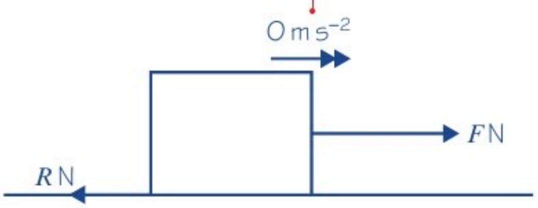
$= 105$

The work done by the force is 105 J.

Use F for force and s for distance.

Example 2

A packing case is pulled across a horizontal floor by a horizontal rope. The case moves at a constant speed and there is a constant resistance to motion of magnitude R newtons. When the case has moved a distance of 12 m the work done is 96 J.
Calculate the magnitude of the resistance.



Work done $= Fs$

$96 = F \times 12$

$F = 8$

$F - R = 0$

$8 - R = 0$

$R = 8$

The magnitude of the resistance is 8 N.

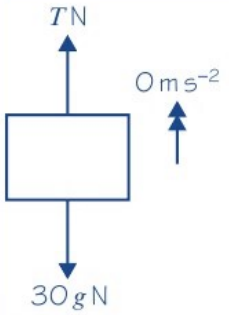
The case moves at a constant speed so its acceleration is $0\,\text{m s}^{-2}$.

Use work done $= Fs$ to calculate the magnitude of the horizontal force.

Use $F = ma$ to calculate the value of R.

Example 3

A bricklayer raises a load of bricks of total mass 30 kg at a constant speed by attaching a cable to the bricks. Assuming the cable is vertical, calculate the work done when the bricks are raised a distance of 7 m.



$T - 30g = 0$

$T = 30g$

work done $= Fs$

$= 30 \times 9.8 \times 7$

$= 2058$

The work done against gravity is 2100 J or 2.1 kJ (2 s.f.)

Use $F = ma$ to calculate the tension in the cable.

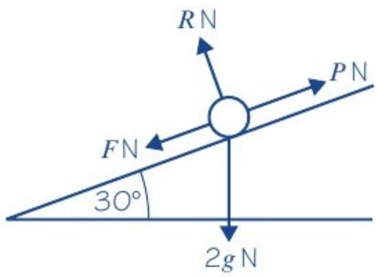
This is $30g$ × distance raised.
You could also use work done against gravity $= mgh$ for this question.

Example 4

A package of mass 2 kg is pulled at a constant speed up a rough plane which is inclined at 30° to the horizontal. The coefficient of friction between the package and the surface is 0.35.
The package is pulled 12 m up a line of greatest slope of the plane. Calculate:

a the work done against gravity **b** the work done against friction

a



Work done against gravity $= mgh$

$= 2g \times 12 \sin 30°$

$= 2 \times 9.8 \times 12 \times 0.5$

$= 117.6$

The work done against gravity is 118 J (3 s.f.).

When the package moves 12 m along the plane, the change in vertical height is 12 sin 30° m.

b Resolve perpendicular to the plane to find R.

$R - 2g\cos 30° = 0$

$R = 2g\cos 30°$

$F = \mu R$

$F = 0.35 \times 2g\cos 30°$

$F - P = 0$

$P = F$

Work done against friction $= Ps$

$= (0.35 \times 2g\cos 30°) \times 12$

$= 71.29\ldots$

The work done against friction is 71.3 J (3 s.f.).

There is no motion perpendicular to the plane.

The particle is moving so friction is limiting.

Watch out You will usually be allowed to give answers to either 2 s.f. or 3 s.f., but read questions carefully, as sometimes a degree of accuracy may be specified.

Example 5

A sledge is pulled 15 m across a smooth sheet of ice by a force of magnitude 27 N. The force is inclined at 25° to the horizontal. By modelling the sledge as a particle calculate the work done by the force.

Work done = horizontal component of force × distance moved

= 27 cos 25° × 15

= 367.0…

The work done by the force is 367 J (3 s.f.).

Watch out The point of application of the force is not moving in the direction of the line of action of the force. To find the work done, use the horizontal component of the force.

Exercise 2A

Whenever a numerical value of g is required, take $g = 9.8\,\text{m}\,\text{s}^{-2}$, and give your answers to either 2 significant figures or 3 significant figures.

1 Calculate the work done by a horizontal force of magnitude 0.6 N which pulls a particle a distance of 4.2 m across a horizontal floor.

2 A box is pulled 12 m across a smooth horizontal floor by a constant horizontal force. The work done by the force is 102 J. Calculate the magnitude of the force.

3 Calculate the work done against gravity when a particle of mass 0.35 kg is raised a vertical distance of 7 m.

4 A crate of mass 15 kg is raised through a vertical distance of 4 m. Calculate the work done against gravity.

5 A box is pushed 15 m across a horizontal surface. The box moves at a constant speed and the resistances to motion are constant and total 22 N. Calculate the work done by the force pushing the box.

6 A ball of mass 0.5 kg falls vertically 15 m from rest. Calculate the work done by gravity.

7 A cable is attached to a crate of mass 80 kg. The crate is raised vertically at a constant speed from the ground to the top of a building. The work done in raising the crate is 30 kJ. Calculate the height of the building.

8 A sledge is pulled 14 m across a horizontal sheet of ice by a rope inclined at 25° to the horizontal. The tension in the rope is 18 N and the ice can be assumed to be a smooth surface.

a Calculate the work done.

b State a modelling assumption that you have made and assess its validity.

9 A parcel of mass 3 kg is pulled a distance of 4 m across a rough horizontal floor. The parcel moves at a constant speed. The work done against friction is 30 J. Calculate the coefficient of friction between the parcel and the surface.

(P) **10** A block of wood of mass 2 kg is pushed across a rough horizontal floor. The block moves at $3\,\text{m}\,\text{s}^{-1}$ and the coefficient of friction between the block and the floor is 0.55. Calculate the work done in 2 seconds.

11 A girl of mass 52 kg climbs a vertical cliff which is 46 m high. Calculate the work she does against gravity.

12 A child of mass 25 kg slides 2 m down a smooth slope inclined at 35° to the horizontal. Calculate the work done by gravity.

13 A particle of mass 0.3 kg is pulled 2 m up a line of greatest slope of a plane which is inclined at 25° to the horizontal. Calculate the work done against gravity.

(E/P) **14** A rough plane surface is inclined at an angle α to the horizontal, where $\sin\alpha = \frac{5}{13}$. A packet of mass 8 kg is pulled at a constant speed up a line of greatest slope of the plane. The coefficient of friction between the packet and the plane is 0.3.

Problem-solving

Draw a right-angled triangle with sides of length 5, 12 and 13, or use your calculator to find the exact value of $\cos\alpha$.

a Calculate the magnitude of the frictional force acting on the packet. **(5 marks)**

The packet moves a distance of 15 m up the plane. Calculate:

b the work done against friction **(3 marks)**

c the work done against gravity. **(4 marks)**

(E/P) **15** A rough surface is inclined at an angle $\arcsin \frac{7}{25}$ to the horizontal. A particle of mass 0.5 kg is pulled 3 m at a constant speed up the surface by a force acting along a line of greatest slope. The only resistances to the motion are those due to friction and gravity. The work done by the force is 12 J. Calculate the coefficient of friction between the particle and the surface. **(5 marks)**

(E) **16** A rough surface is inclined at 40° to the horizontal. A box of mass 1.5 kg is pulled at a constant speed up the surface by a force T acting along a line of greatest slope. The coefficient of friction between the particle and the surface is 0.4.

Modelling the box as a particle, calculate the work done by T when the particle travels 8 m. **(4 marks)**

(E) **17** A particle P of mass 2 kg is projected up a line of greatest slope of a rough plane which is inclined at an angle $\arcsin \frac{3}{5}$ to the horizontal. The coefficient of friction between P and the plane is 0.35.

The particle comes to instantaneous rest a distance 3 m up the line of greatest slope of the plane. Find:

a the work done by gravity **(3 marks)**

b the work done by friction **(3 marks)**

c the speed of projection. **(4 marks)**

2.2 Kinetic and potential energy

You can calculate the **kinetic energy** of a moving particle and the **potential energy** of a particle.

- **Kinetic energy (K.E.) = $\frac{1}{2}mv^2$, where m is the mass of the particle and v is its speed.**
- **Potential energy (P.E.) = mgh, where h is the height of the particle above an arbitrary fixed level.**

Note A level candidates might need to consider the kinetic energy of a particle moving in two dimensions. If a particle has mass m and velocity vector **v** then it has kinetic energy $\frac{1}{2}m|\mathbf{v}|^2$.

A particle possesses kinetic energy when it is moving. When the mass of the particle is measured in kilograms and its velocity is measured in metres per second, the kinetic energy is measured in joules.

A particle possesses potential energy whenever gravity acts on it. When the mass of the particle is measured in kilograms, the acceleration due to gravity is measured in metres per second squared and its height above the fixed level is measured in metres, the potential energy is measured in joules.

Notation This type of potential energy is sometimes called **gravitational potential energy**.

Watch out Kinetic energy cannot be negative. Potential energy is negative if the particle is below the fixed reference level.

The work done by a force which accelerates a particle horizontally is related to the kinetic energy of that particle.

- **Work done = change in kinetic energy**

You can derive this result using formulae you already know:

$F = ma$ — Equation of motion

$v^2 = u^2 + 2as$ — Constant acceleration formula

$a = \frac{v^2 - u^2}{2s}$ — Rearrange to make a the subject

$F = \frac{m(v^2 - u^2)}{2s}$ — Substitute for a in the equation of motion.

$Fs = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$ — Fs is force × distance or work done

Work done = final K.E. – initial K.E.

Work done = change in K.E.

Example 6

A particle of mass 0.3 kg is moving at a speed of 9 m s^{-1}

Calculate its kinetic energy.

K.E. $= \frac{1}{2}mv^2$

$= \frac{1}{2} \times 0.3 \times 9^2$

$= 12.15$

The K.E. of the particle is 12.2 J (3 s.f.)

Example 7

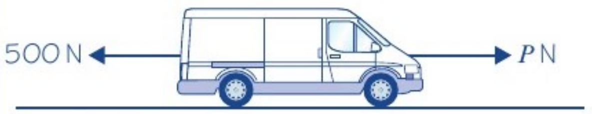
A box of mass 1.5 kg is pulled across a smooth horizontal surface by a horizontal force. The initial speed of the box is $u\,\text{m s}^{-1}$ and its final speed is $3\,\text{m s}^{-1}$. The work done by the force is 1.8 J. Calculate the value of u.

$$\text{Work done} = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2$$
$$1.8 = \tfrac{1}{2} \times 1.5 \times 3^2 - \tfrac{1}{2} \times 1.5u^2$$
$$\tfrac{1}{2} \times 1.5u^2 = 4.95$$
$$u^2 = \frac{4.95 \times 2}{1.5}$$
$$u = 2.57 \text{ (3 s.f.)}$$

Watch out The work done by the force is equal to the **change** in kinetic energy.

Example 8

A van of mass 2000 kg starts from rest at some traffic lights. After travelling 400 m the van's speed is $12\,\text{m s}^{-1}$. A constant resistance of 500 N acts on the van. Calculate the driving force, which can be assumed to be constant.



Work done = increase in K.E.

$$Fs = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2$$
$$(P - 500) \times 400 = \tfrac{1}{2} \times 2000 \times 12^2 - \tfrac{1}{2} \times 2000 \times 0^2$$
$$P - 500 = \frac{\tfrac{1}{2} \times 2000 \times 12^2}{400}$$
$$P = 360 + 500$$
$$P = 860$$

The driving force is 860 N.

The force used to calculate the work done is the resultant force in the direction of the motion.

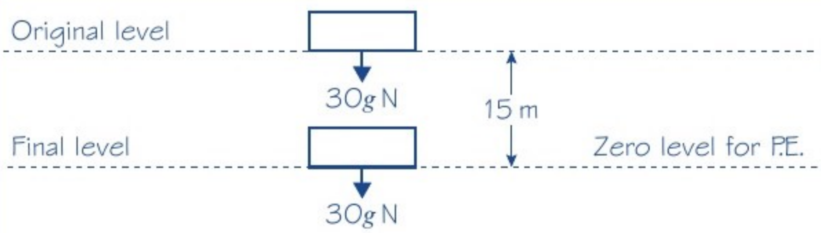
Watch out When you are considering an overall change in energy of a body, you should use the resultant force in your calculations. The change in energy of the van is not the same as the work done by the driving force, as some of this work is used to overcome friction.

- **You must choose a zero level of potential energy before calculating a particle's potential energy.**

If the particle moves upwards its potential energy will increase. If it moves downwards its potential energy will decrease.

Example 9

A load of bricks of total mass 30 kg is lowered vertically to the ground through a distance of 15 m. Find the loss in potential energy.

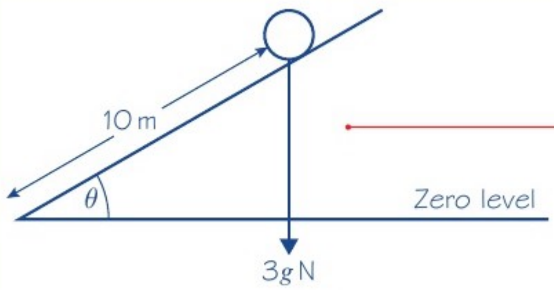


Final P.E. = 0
Initial P.E. = mgh
= 30 × 9.8 × 15
= 4410
Loss of P.E. = 4410 − 0
The loss of potential energy is 4410 J.

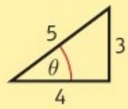
Take the final level to be the zero level for P.E.

Example 10

A parcel of mass 3 kg is pulled 10 m up a plane inclined at an angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$. Assuming that the parcel moves up a line of greatest slope of the plane, calculate the potential energy gained by the parcel.



$\tan\theta = \frac{3}{4}$ so $\sin\theta = \frac{3}{5}$



Change in height = $10\sin\theta$
= 6 m
Final P.E. = mgh
= 3 × 9.8 × 6
= 176.4
Initial P.E. = 0
The potential energy gained by the parcel is 176 J.

The vertical distance moved by the parcel is 6 m.

Exercise 2B

Whenever a numerical value of g is required, take $g = 9.8\,\text{m}\,\text{s}^{-2}$, and give your answers to either 2 significant figures or 3 significant figures.

1 Calculate the kinetic energy of the following objects. Put them in order, from the greatest kinetic energy to the least.

a a particle of mass 0.3 kg moving at $15\,\text{m}\,\text{s}^{-1}$

b a particle of mass 3 kg moving at $2\,\text{m}\,\text{s}^{-1}$

c an arrow of mass 0.1 kg moving at $100\,\text{m}\,\text{s}^{-1}$

d a boy of mass 25 kg running at $4\,\text{m}\,\text{s}^{-1}$

e a car of mass 800 kg moving at $20\,\text{m}\,\text{s}^{-1}$

2 Find the change in potential energy of each of the following, stating in each case whether it is a loss or a gain:

a a particle of mass 1.5 kg raised through a vertical distance of 3 m

b a woman of mass 55 kg ascending a vertical distance of 15 m

c a man of mass 75 kg descending a vertical distance of 30 m

d a lift of mass 580 kg descending a vertical distance of 6 m.

3 A particle of mass 1.2 kg decreases its speed from $12\,\text{m s}^{-1}$ to $4\,\text{m s}^{-1}$. Calculate the decrease in the particle's kinetic energy.

4 A van of mass 900 kg increases its speed from $5\,\text{m s}^{-1}$ to $20\,\text{m s}^{-1}$. Calculate the increase in the van's kinetic energy.

(P) **5** A particle of mass 0.2 kg increases its speed from $2\,\text{m s}^{-1}$ to $v\,\text{m s}^{-1}$. The particle's kinetic energy increases by 6 J. Calculate the value of v.

(P) **6** An ice skater of mass 45 kg is initially moving at $5\,\text{m s}^{-1}$. She decreases her kinetic energy by 100 J. Calculate her final speed.

(E) **7** A playground slide is modelled as a plane inclined at 48° to the horizontal. A child of mass 25 kg slides down the slide for 4 m.

a Calculate the potential energy lost by the child. **(4 marks)**

b State one assumption that you have made in your calculations, and comment on its validity. **(1 mark)**

(E/P) **8** A ball of mass 0.6 kg is dropped from a height of 2 m into a pond.

a Calculate the kinetic energy of the ball as it hits the surface of the water. **(3 marks)**

The ball begins to sink in the water with a speed of $4.8\,\text{m s}^{-1}$.

b Calculate the kinetic energy lost when the ball strikes the water. **(4 marks)**

9 A lorry of mass 2000 kg is initially travelling at $35\,\text{m s}^{-1}$. The brakes are applied, causing the lorry to decelerate at $1.2\,\text{m s}^{-2}$ for 5 s. Calculate the loss of kinetic energy of the lorry.

(E) **10** A car of mass 750 kg moves along a stretch of road which can be modelled as a line of greatest slope of a plane inclined to the horizontal at 30°. As the car moves up the road for 500 m its speed reduces from $20\,\text{m s}^{-1}$ to $15\,\text{m s}^{-1}$. Calculate:

a the loss of kinetic energy of the car **(3 marks)**

b the gain of potential energy of the car. **(4 marks)**

(E/P) **11** A woman of mass 80 kg climbs a vertical cliff face of height h m. Her potential energy increases by 15.7 kJ.

Find the height of the cliff. **(3 marks)**

Challenge

A 1 kg ball, initially at rest, is dropped from the top of a cliff.
The ball can be modelled as a particle falling freely under gravity.

a Find, in terms of t, the kinetic and potential energy of the ball at a time t seconds after it is dropped.

b Show that the sum of the kinetic energy and potential energy of the ball is constant.

2.3 Conservation of mechanical energy and work–energy principle

You can use the principle of conservation of mechanical energy and the work–energy principle to solve problems involving a moving particle.

- **When no external forces (other than gravity) do work on a particle during its motion, the sum of the particle's kinetic energy and potential energy remains constant.**

Notation This is called the **principle of conservation of mechanical energy**.

This is true whether the particle moves vertically or along a path inclined to the horizontal.

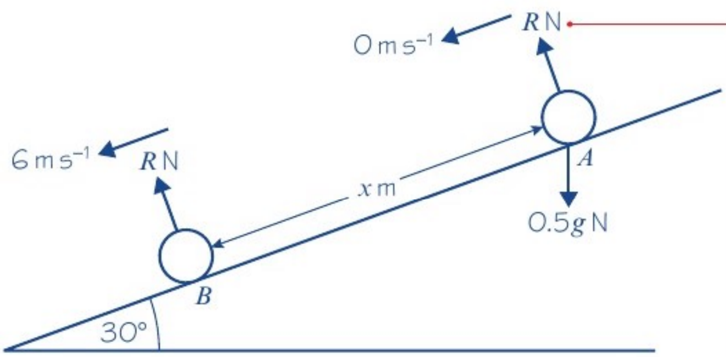
The total energy possessed by a particle can only change during the particle's motion if some external force is doing work on the particle. Any non-gravitational resistance to motion acting on the particle will reduce the total energy of the particle, as the particle will have to do work to overcome the resistance.

- **The change in the total energy of a particle is equal to the work done on the particle.**

Notation This is called the **work–energy principle**.

Example 11

A smooth plane is inclined at 30° to the horizontal. A particle of mass 0.5 kg slides down a line of greatest slope of the plane. The particle starts from rest at point A and passes point B with a speed $6\,\text{m s}^{-1}$. Find the distance AB.



Note that the normal reaction R does no work on the particle as it is always perpendicular to the motion.

Decrease in P.E. $= mgh$

$= 0.5 \times 9.8 \times (x \sin 30°)$

The vertical distance moved by the particle is $x \sin 30°$, where x is the distance AB.

Increase in K.E. $= \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$= \frac{1}{2} \times 0.5 \times 6^2 - 0$

Decrease in P.E. = increase in K.E.

$0.5 \times 9.8 \times (x \sin 30°) = \frac{1}{2} \times 0.5 \times 6^2$

$$x = \frac{\frac{1}{2} \times 0.5 \times 6^2}{0.5 \times 9.8 \times \sin 30°}$$

$x = 3.673...$

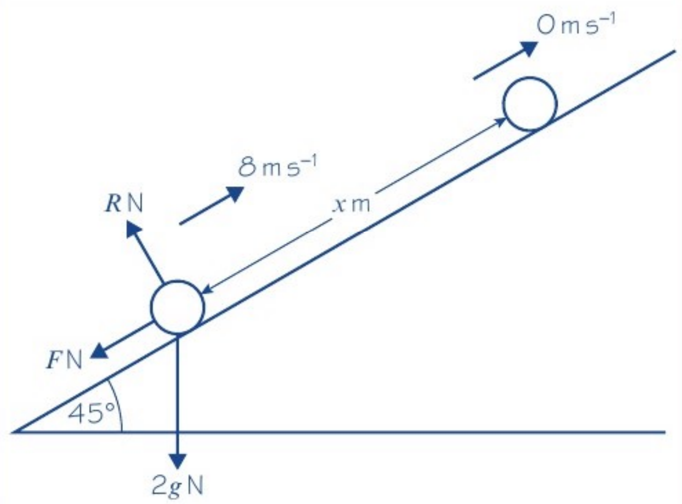
The distance AB is 3.67 m (3 s.f.)

Problem-solving

The only force acting on the particle that is doing work is gravity so you can apply the principle of conservation of mechanical energy.

Example 12

A particle of mass 2 kg is projected with speed $8\,\text{m s}^{-1}$ up a line of greatest slope of a rough plane inclined at 45° to the horizontal. The coefficient of friction between the particle and the plane is 0.4. Calculate the distance the particle travels up the plane before coming to instantaneous rest.



Total loss of energy = K.E. lost − P.E. gained

$= \left(\frac{1}{2}mv^2 - \frac{1}{2}mu^2\right) - mgh$

$= \left(\frac{1}{2} \times 2 \times 8^2 - 0\right) - 2 \times 9.8 \times (x \sin 45°)$

$= 64 - 19.6x \sin 45°$

Work done against friction = Fx

$R = 2g \cos 45°$

$F = \mu R = 0.4 \times 2g \cos 45° = 0.8g \cos 45°$

Loss of energy = work done against friction

$64 - 19.6x \sin 45° = 0.8g \cos 45° \times x$

$19.6x \sin 45° + 0.8gx \cos 45° = 64$

$x = \frac{64}{19.6 \sin 45° + 0.8g \cos 45°} = 3.298...$

The particle moves 3.30 m (3 s.f.) up the plane.

Problem-solving

The slope is rough so some work will be done on the particle by a force other than gravity. This means you will have to use the work-energy principle in this question.

The particle has lost energy through having to work to overcome the frictional force.

You need to find R so you can find F. Resolve perpendicular to the plane to find R.

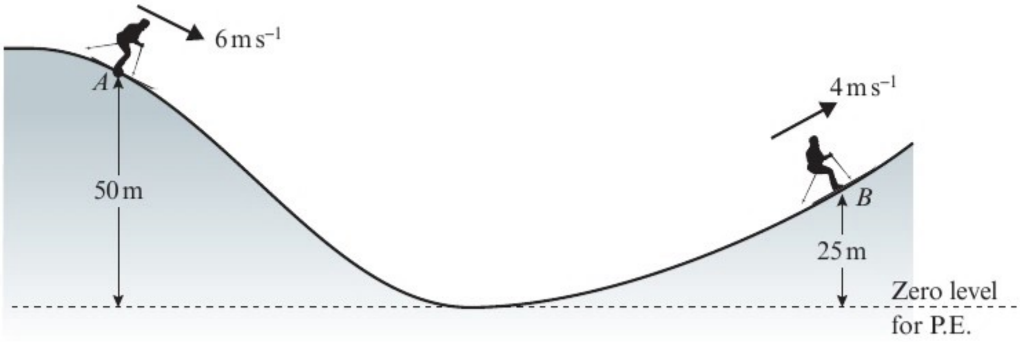
The particle is moving so friction is limiting.

This is because of the work–energy principle.

Example 13

A skier moving downhill passes point A on a ski run at $6\,\text{m s}^{-1}$. After descending 50 m vertically the run begins to ascend. When the skier has ascended 25 m to point B her speed is $4\,\text{m s}^{-1}$.

The skier and her skis have a combined mass of 55 kg. The total distance she travels from A to B is 1400 m. The non-gravitational resistances to motion are constant and have a total magnitude of 12 N. Calculate the work done by the skier.



Loss of K.E. $= \frac{1}{2}mu^2 - \frac{1}{2}mv^2$
$= \frac{1}{2} \times 55 \times 6^2 - \frac{1}{2} \times 55 \times 4^2$
$= 550$ J

Her final speed is less than her initial speed so K.E. is lost.

Loss of P.E. $= mgh$
$= 55 \times 9.8 \times (50 - 25)$
$= 13\,475$ J

B is lower than *A* so P.E. is lost.

Total loss of energy = 550 + 13 475 = 14 025 J

Work done against resistances = 12 × 1400 = 16 800 J

Use work done = *Fs* to calculate the work done against the resistances.

Work done by skier = 16 800 − 14 025 = 2775 J

The skier's loss of energy provides some of the work needed to overcome the resistances. The remainder is provided by the skier doing work.

The work done by the skier is 2780 J (3 s.f.)

Exercise 2C

Whenever a numerical value of *g* is required, take $g = 9.8\,\text{m s}^{-2}$, and give your answers to either 2 significant figures or 3 significant figures.

Hint Where possible, you should use the principle of conservation of mechanical energy and the work-energy principle to answer these questions rather than the *suvat* formulae.

1 A particle of mass 0.4 kg falls a vertical distance of 7 m from rest.
- **a** Calculate the potential energy lost.
- **b** By assuming that air resistance can be neglected, calculate the final speed of the particle.

2 A stone of mass 0.5 kg is dropped from the top of a tower and falls vertically to the ground. It hits the ground with a speed of $12\,\text{m s}^{-1}$. Find:
- **a** the kinetic energy gained by the stone
- **b** the potential energy lost by the stone
- **c** the height of the tower.

3 A box of mass 6 kg is pulled in a straight line across a smooth horizontal floor by a constant horizontal force of magnitude 10 N. The box has speed $2.5\,\text{m s}^{-1}$ when it passes through point *P* and speed $5\,\text{m s}^{-1}$ when it passes through point *Q*.
- **a** Find the increase in kinetic energy of the box.
- **b** Write down the work done by the force.
- **c** Find the distance *PQ*.

4 A particle of mass 0.4 kg moves in a straight line across a rough horizontal surface. The speed of the particle decreases from $8\,\text{m}\,\text{s}^{-1}$ to $4\,\text{m}\,\text{s}^{-1}$ as it travels 7 m.

a Calculate the kinetic energy lost by the particle.

b Write down the work done against friction.

c Calculate the coefficient of friction between the particle and the surface.

E **5** A box of mass 3 kg is projected from point A across a rough horizontal floor with speed $6\,\text{m}\,\text{s}^{-1}$. The box moves in a straight line across the floor and comes to rest at point B. The coefficient of friction between the box and the floor is 0.4.

a Calculate the kinetic energy lost by the box. **(2 marks)**

b Write down the work done against friction. **(1 mark)**

c Calculate the distance AB. **(5 marks)**

6 A particle of mass 0.8 kg falls a vertical distance of 5 m from rest. By considering energy, find the speed of the particle as it hits the ground.

7 A stone of mass 0.3 kg is dropped from the top of a vertical cliff and falls freely under gravity. It hits the ground below with a speed of $20\,\text{m}\,\text{s}^{-1}$. Air resistance is negligible. Use energy considerations to calculate the height of the cliff.

8 A particle of mass 0.3 kg is projected vertically upwards and moves freely under gravity. The initial speed of the particle is $u\,\text{m}\,\text{s}^{-1}$. When the particle is 5 m above the point of projection its kinetic energy is 2.1 J. Neglecting air resistance, calculate the value of u.

9 A bullet of mass 0.1 kg travelling at $500\,\text{m}\,\text{s}^{-1}$ horizontally hits a vertical wall. The bullet penetrates the wall to a depth of 50 mm. The resistive force exerted on the bullet by the wall is constant. Calculate the magnitude of the resistive force.

E **10** A bullet of mass 150 g travelling at $500\,\text{m}\,\text{s}^{-1}$ horizontally hits a vertical wall. The resistive force exerted by the wall on the bullet is modelled as having a constant magnitude of 250 000 N.

a Calculate the distance that the bullet penetrates the wall. **(4 marks)**

b State how the model for the resistive force could be refined to make it more realistic. **(1 mark)**

11 A package of mass 5 kg is released from rest and slides 2 m down a line of greatest slope of a smooth plane inclined at 35° to the horizontal.

a Calculate the potential energy lost by the package.

b Write down the kinetic energy gained by the package.

c Calculate the final speed of the package.

12 A particle of mass 0.5 kg is released from rest and slides down a line of greatest slope of a smooth plane inclined at 30° to the horizontal. When the particle has moved a distance x m, its speed is $2\,\text{m}\,\text{s}^{-1}$. Find the value of x.

13 A particle of mass 0.2 kg is projected with speed $9\,\text{m}\,\text{s}^{-1}$ up a line of greatest slope of a smooth plane inclined at 30° to the horizontal. The particle travels a distance x m before first coming to rest. By considering energy, calculate the value of x.

14 A particle of mass 0.6 kg is projected up a line of greatest slope of a smooth plane inclined at 40° to the horizontal. The particle travels 5 m before first coming to rest. Use energy considerations to calculate the speed of projection.

E/P **15** A box of mass 2 kg is projected with speed $6\,\mathrm{m\,s^{-1}}$ up a line of greatest slope of a rough plane inclined at 30° to the horizontal. The coefficient of friction between the box and the plane is $\frac{1}{3}$. Use the work–energy principle to calculate the distance the box travels up the plane before coming to rest. **(5 marks)**

E **16** A tennis ball of mass 58 g is hit vertically upwards by a tennis racket from a point 1 metre above the ground. Given that the tennis ball receives an impulse of 1.7 N s,

a find the initial speed of the ball. **(3 marks)**

The ball first comes to rest at a point 28 m above the ground.

b Find the energy lost due to air resistance by the ball during this motion. **(3 marks)**

Given that the air resistance is modelled as a constant force of magnitude R N,

c find the value of R. **(2 marks)**

E/P **17** A skier of mass 80 kg skis down a straight hill inclined at 30° to the horizontal. The speed of the skier increases from $3\,\mathrm{m\,s^{-1}}$ to $12\,\mathrm{m\,s^{-1}}$.

The total resistances to motion of the skier due to friction and air resistance are modelled as a constant force of magnitude R N.

a Given that the skier travels a total distance of 50 m, find the value of R. **(5 marks)**

b Suggest one way in which the model could be refined. **(1 mark)**

E/P **18** A box of mass 70 kg starts at rest and slides in a straight line down a surface inclined at 20° to the horizontal. After the box has travelled a distance of 60 m, the surface becomes horizontal, and the box slides a further 50 m before coming to rest. The total resistance due to friction and air resistance is modelled as a force of constant magnitude R N which acts so as to oppose the direction of motion of the box. Find the value of R. **(7 marks)**

E/P **19** A girl and her sledge have a combined mass of 40 kg. She starts from rest and descends a slope which is inclined at 25° to the horizontal. At the bottom of the slope the ground becomes horizontal for 15 m before rising at 6° to the horizontal. The girl travels 25 m up the slope before coming to rest once more. There is a constant resistance to motion of magnitude 18 N. Calculate the distance the girl travels down the slope initially. **(7 marks)**

Challenge

The temperature of a gas is related to the average kinetic energy of its molecules by the formula:

average K.E. = $\frac{3}{2}kT$

where $k = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ and T is the temperature in kelvin (K).

The mass of an oxygen molecule is 8 times greater than the mass of a hydrogen molecule. Two containers, one containing hydrogen and the other containing oxygen, have been in contact and able to exchange heat for a very long time, so that molecules of both gases are at the same temperature. The average speed of the oxygen molecules is $400\,\mathrm{m\,s^{-1}}$. Find the average speed of the hydrogen molecules.

2.4 Power

You can calculate the power developed by an engine and solve problems about moving vehicles.

- **Power is the rate of doing work.**

The power developed by the engine of a moving vehicle is calculated using the following formula.

- **Power = Fv, where F is the driving force produced by the engine and v is the speed of the vehicle.**

If the driving force is measured in newtons and the speed is measured in m s^{-1}, power is measured in watts (W), where **1 watt is 1 joule per second**. The power of an engine is often given in kilowatts (kW).

Example 14

A truck is being pulled up a slope at a constant speed of $8\,\text{m s}^{-1}$ by a force of magnitude 2000 N acting parallel to the direction of motion of the truck. Calculate, in kilowatts, the power developed.

Work done per second = 2000 × 8 = 16 000 J

Work done per second = force × distance moved per second

Power = rate of doing work = 16 000 W

The power developed is 16 kW.

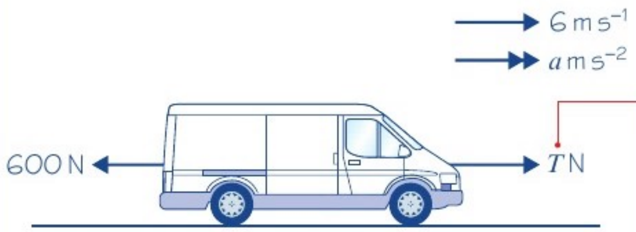
1 kW = 1000 W

Example 15

A van of mass 1250 kg is travelling along a horizontal road. The van's engine is working at 24 kW. The constant resistance to motion has magnitude 600 N. Calculate:

a the acceleration of the van when it is travelling at $6\,\text{m s}^{-1}$

b the maximum speed of the van.



We often use T for the tractive (or pulling) force.

a Power = 24 kW = 24 000 W

You must work with power in watts.

Power = Fv

Use power = Fv to find the driving force.

$24\,000 = T \times 6$

$T = 4000$

$4000 - 600 = 1250a$

Use $F = ma$ to find the acceleration.

$$a = \frac{4000 - 600}{1250} = 2.72$$

The acceleration is $2.72\,\text{m s}^{-2}$

The tractive force will be different in part **b**. You can use a different letter, or use T' (pronounced ***T* prime**).

b $T' = 600\text{ N}$

At maximum speed there will be no acceleration, so the resultant horizontal force will be zero.

$24\,000 = 600v$

Use power = Fv to find the speed.

$$v = \frac{24\,000}{600} = 40$$

The maximum speed of the van is $40\,\text{m s}^{-1}$.

Example 16

A car of mass 1100 kg is travelling at a constant speed of $15\,\text{m s}^{-1}$ along a straight road which is inclined at 7° to the horizontal. The engine is working at a rate of 24 kW.

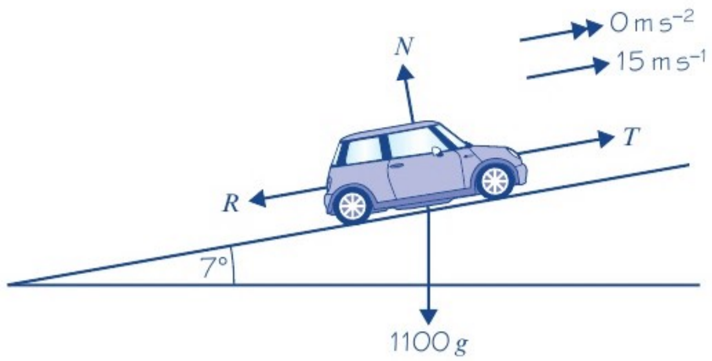
a Calculate the magnitude of the non-gravitational resistance to motion.

The rate of working of the engine is now increased to 28 kW.

Assuming the resistances to motion are unchanged,

b calculate the initial acceleration of the car.

a



$24 \times 10^3 = T \times 15$ — Use power $= Fv$ to find T.

$T = \dfrac{24 \times 10^3}{15} = 1600$

$R + 1100g\sin 7° = T = 1600$ — Resolve along the slope to find the resistance.

$R = 1600 - 1100g\sin 7°$

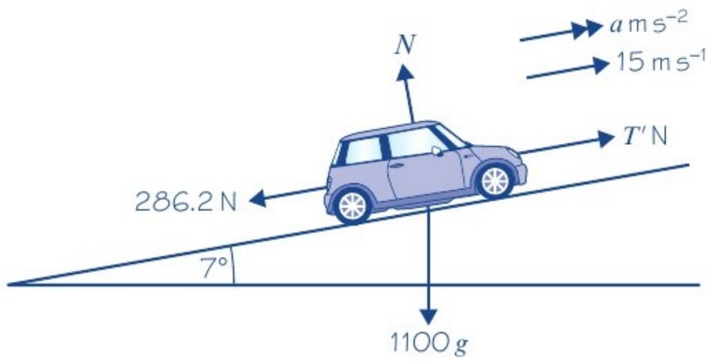
$R = 286.2\ldots$

The resistance to motion is 286 N (3 s.f.)

Problem-solving

Draw a new diagram for the new situation. As the power has changed, the driving force and the acceleration will change.

b



$28 \times 10^3 = T' \times 15$ — Use power $= Fv$ to find the new driving force. Initially, the speed will be $15\,\text{m s}^{-1}$

$T' = \dfrac{28 \times 10^3}{15}$

$T' - (286.2 + 1100g\sin 7°) = 1100a$ — Use $F = ma$ to find the acceleration. The more significant figures you use when carrying through previously calculated answers, the more accurate your final answer will be.

$(28 \times 10^3) \div 15 - (286.2 + 1100g\sin 7°) = 1100a$

$a = \dfrac{(28 \times 10^3) \div 15 - (286.2 + 1100g\sin 7°)}{1100}$

$a = 0.2424\ldots$

The initial acceleration is $0.242\,\text{m s}^{-2}$ (3 s.f.). — We need 'initial' here as once the car accelerates its speed increases so either the power or the driving force will change.

Example 17

A car of mass 2600 kg is travelling in a straight line. At the instant when the speed of the car is $v\,\text{m s}^{-1}$, the total resistances to motion are modelled as a variable force of magnitude $(800 + 5v^2)$ N. The car has a cruise control feature which adjusts the power generated by the engine to maintain a constant speed of $18\,\text{m s}^{-1}$.

Find the power generated by the engine when:

a the car is travelling on a horizontal road

b the car is travelling up a road that is inclined at an angle 4° to the horizontal.

a When $v = 18\,\text{m s}^{-1}$,

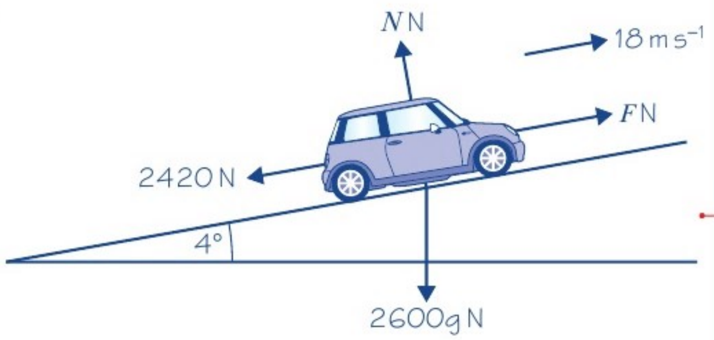
resistive force = $(800 + 5 \times 18^2) = 2420$ N

$P = Fv$

$P = 2420 \times 18 = 43\,560$ W

The power generated by the engine is 43 600 W (3 s.f.)

b



$F = 2420 + 2600g\sin 4°$

$= 4197$ N

$P = 4197 \times 18 = 75\,553$ W

The power generated by the engine is 75 600 W (3 s.f.).

Problem-solving

The magnitude of the resistive force is **variable**. However, the car is maintaining a constant speed, so substitute $v = 18$ into $800 + 5v^2$ to find the magnitude of the resistive force.

The car is travelling at a constant speed so the driving force provided by the engine is equal to the resistive force.

The magnitude of the resistive force is the same.

Resolve parallel to the plane.

Substitute into power = force × velocity

Exercise 2D

Whenever a numerical value of g is required, take $g = 9.8\,\text{m s}^{-2}$, and give your answers to either 2 significant figures or 3 significant figures.

1. A force of 1500 N pulls a van up a slope at a constant speed of $12\,\text{m s}^{-1}$. Calculate, in kW, the power developed.

2. A car is travelling at $15\,\text{m s}^{-1}$ and its engine is producing a driving force of 1000 N. Calculate the power developed.

3. The engine of a van is working at 5 kW and the van is travelling at $18\,\text{m s}^{-1}$. Find the magnitude of the driving force produced by the van's engine.

4. A car's engine is working at 15 kW. The car is travelling along a horizontal road. The total resistance to motion has a magnitude of 600 N. Calculate the maximum speed of the car.

5 A car has a maximum speed of $40\,\text{m}\,\text{s}^{-1}$ when travelling along a horizontal road. The total resistances to motion of the car are assumed to be constant and of magnitude 500 N.

a Calculate the power the car's engine must develop to maintain this speed.

b Comment on the assumption that the resistance to motion is constant.

6 A van is travelling along a horizontal road at a constant speed of $16\,\text{m}\,\text{s}^{-1}$. The van's engine is working at 8.8 kW. Calculate the magnitude of the resistance to motion.

(E) 7 A car of mass 850 kg is travelling along a straight horizontal road against resistances totalling 350 N. The car's engine is working at 9 kW. Calculate:

a the acceleration when the car is travelling at $7\,\text{m}\,\text{s}^{-1}$ **(3 marks)**

b the acceleration when the car is travelling at $15\,\text{m}\,\text{s}^{-1}$ **(2 marks)**

c the maximum speed of the car. **(2 marks)**

8 A car of mass 900 kg is travelling along a straight horizontal road at a speed of $20\,\text{m}\,\text{s}^{-1}$. The constant resistances to motion total 300 N. The car is accelerating at $0.3\,\text{m}\,\text{s}^{-2}$. Calculate the power developed by the engine.

(E) 9 A car of mass 1000 kg is travelling along a straight horizontal road. The car's engine is working at 12 kW. When its speed is $24\,\text{m}\,\text{s}^{-1}$ its acceleration is $0.2\,\text{m}\,\text{s}^{-2}$. The resistances to motion have a total magnitude of R newtons. Calculate the value of R. **(4 marks)**

10 A cyclist is travelling along a straight horizontal road. The resistance to his motion is constant and has magnitude 28 N. The maximum rate at which he can work is 280 W. Calculate his maximum speed.

(E) 11 A van of mass 1200 kg is travelling up a straight road inclined at 5° to the horizontal. The van moves at a constant speed of $20\,\text{m}\,\text{s}^{-1}$ and its engine is working at 24 kW. The resistance to motion from non-gravitational forces has magnitude R newtons.

a Calculate the value of R. **(3 marks)**

The road now becomes horizontal. The resistance to motion from non-gravitational forces is unchanged.

b Calculate the initial acceleration of the van. **(4 marks)**

(E) 12 A car of mass 800 kg is travelling at $18\,\text{m}\,\text{s}^{-1}$ along a straight horizontal road. The car's engine is working at a constant rate of 26 kW against a constant resistance of magnitude 750 N.

a Find the acceleration of the car. **(3 marks)**

The car now ascends a straight hill, inclined at 9° to the horizontal. The resistance to motion from non-gravitational forces is unchanged and the car's engine works at the same rate.

b Find the maximum speed at which the car can travel up the hill. **(4 marks)**

(E) 13 A van of mass 1500 kg is travelling at its maximum speed of $30\,\text{m}\,\text{s}^{-1}$ along a straight horizontal road against a constant resistance of magnitude 600 N.

a Find the power developed by the van's engine. **(3 marks)**

The van now travels up a hill along a straight road inclined at 8° to the horizontal. The van's engine works at the same rate and the resistance to motion from non-gravitational forces is unchanged.

b Find the maximum speed at which the van can ascend the hill. **(4 marks)**

(P) **14** A cyclist with her bicycle has a total mass of 80 kg. She travels at a constant speed of $7\,\text{m}\,\text{s}^{-1}$, first on flat ground, and then up a hill inclined at 2° to the horizontal. Find the increase in power required on the hill compared to the flat ground.

(E) **15** A train of mass 150 tonnes is moving up a straight track which is inclined at 2° to the horizontal. The resistance to the motion of the train from non-gravitational forces has magnitude 6 kN and the train's engine is working at a constant rate of 350 kW.

a Calculate the maximum speed of the train. **(5 marks)**

The track now becomes horizontal. The engine continues to work at 350 kW and the resistance to motion remains 6 kN.

b Find the initial acceleration of the train. **(3 marks)**

(E/P) **16** A car is moving along a straight horizontal road with speed $v\,\text{m}\,\text{s}^{-1}$. The magnitude of the resistance to motion of the car is given by the formula $(150 + 3v)$ N. The car's engine is working at 10 kW. Calculate the maximum value of v. **(6 marks)**

(E/P) **17** A van of mass 4000 kg is travelling in a straight line. At the instant when the speed of the car is $v\,\text{m}\,\text{s}^{-1}$, the total resistances to motion are modelled as a variable force of magnitude $(1200 + 8v)$ N. The engine of the van works at a constant rate of 28 kW.

Problem-solving

For part **b**, use $P = Fv$ to find the driving force in terms of w.

a Find the acceleration of the van at the instant when $v = 10$. **(4 marks)**

When the car is travelling at $w\,\text{m}\,\text{s}^{-1}$, it is decelerating at $0.2\,\text{m}\,\text{s}^{-2}$.

b Find the value of w. **(4 marks)**

Mixed exercise 2

Whenever a numerical value of g is required, take $g = 9.8\,\text{m}\,\text{s}^{-2}$, and give your answers to either 2 significant figures or 3 significant figures.

1 A cyclist and her bicycle have a combined mass of 70 kg. She is cycling at a constant speed of $6\,\text{m}\,\text{s}^{-1}$ on a straight road up a hill inclined at 5° to the horizontal. She is working at a constant rate of 480 W. Calculate the magnitude of the resistance to motion from non-gravitational forces.

2 A boy hauls a bucket of water through a vertical distance of 25 m. The combined mass of the bucket and water is 12 kg. The bucket starts from rest and finishes at rest.

a Calculate the work done by the boy.

The boy takes 30 s to raise the bucket.

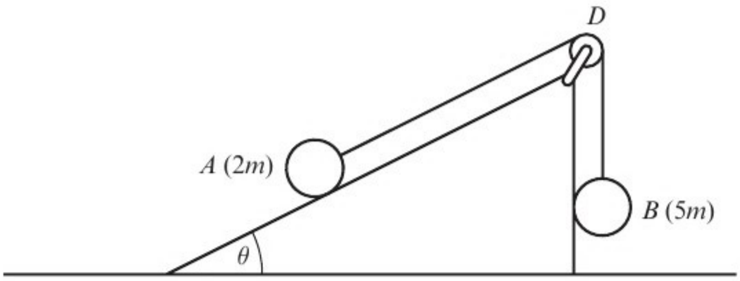
b Calculate the average rate of working of the boy.

(P) **3** A particle P of mass 0.5 kg is moving in a straight line from A to B on a rough horizontal plane. At A the speed of P is $12\,\text{m}\,\text{s}^{-1}$, and at B its speed is $8\,\text{m}\,\text{s}^{-1}$. The distance from A to B is 25 m. The only resistance to motion is the friction between the particle and the plane. Find:

a the work done by friction as P moves from A to B

b the coefficient of friction between the particle and the plane.

E/P **4**



The diagram shows a particle A of mass $2m$ which can move on the rough surface of a plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{3}{5}$. A second particle B of mass $5m$ hangs freely attached to a light inextensible string which passes over a smooth light pulley fixed at D. The other end of the string is attached to A. The coefficient of friction between A and the plane is $\frac{3}{8}$. Particle B is initially hanging 2 m above the ground and A is 4 m from D. When the system is released from rest with the string taut, A moves up a line of greatest slope of the plane.

a Find the initial acceleration of A. **(7 marks)**

When B has descended 1 m the string breaks.

b By using the principle of conservation of energy calculate the total distance moved by A before it first comes to rest. **(5 marks)**

E **5** A car of mass 800 kg is travelling along a straight horizontal road. The resistance to motion from non-gravitational forces has a constant magnitude of 500 N. The engine of the car is working at a rate of 16 kW.

a Calculate the acceleration of the car when its speed is $15\,\text{m}\,\text{s}^{-1}$. **(3 marks)**

The car comes to a hill at the moment when it is travelling at $15\,\text{m}\,\text{s}^{-1}$. The road is still straight but is now inclined at 5° to the horizontal. The resistance to motion from non-gravitational forces is unchanged. The rate of working of the engine is increased to 24 kW.

b Calculate the new acceleration of the car. **(4 marks)**

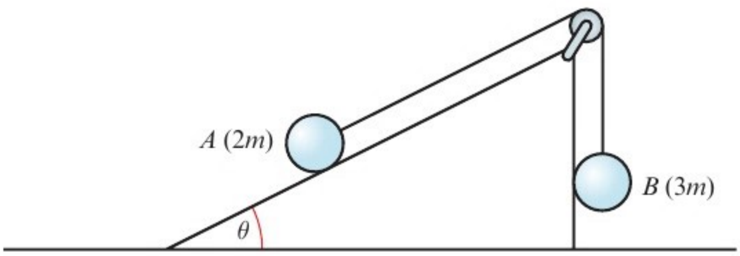
E/P **6** A car of mass 750 kg is moving at a constant speed of $18\,\text{m}\,\text{s}^{-1}$ down a straight road inclined at an angle θ to the horizontal, where $\tan\theta = \frac{1}{20}$. The resistance to motion from non-gravitational forces has a constant magnitude of 1000 N.

a Find, in kW, the rate of working of the car's engine. **(3 marks)**

The engine of the car is now switched off and the car comes to rest t seconds later. The resistance to motion from non-gravitational forces is unchanged.

b Find the value of t. **(3 marks)**

P **7**



The diagram shows a particle A of mass $2m$ which can move on the rough surface of a plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{3}{5}$. A second particle B of mass $3m$ hangs freely attached to a light inextensible string which passes over a smooth pulley.

The other end of the string is attached to A. The coefficient of friction between A and the plane is $\frac{1}{4}$. The system is released from rest with the string taut and A moves up a line of greatest slope of the plane. When each particle has moved a distance s, A has not reached the pulley and B has not reached the ground.

a Find an expression for the potential energy lost by the system when each particle has moved a distance s.

When each particle has moved a distance s they are moving with speed v.

b Find an expression for v^2, in terms of s.

(E) **8** A parcel of mass 5 kg is resting on a platform inclined at 25° to the horizontal. The coefficient of friction between the parcel and the platform is 0.3. The parcel is released from rest and slides down a line of greatest slope of the platform. Calculate:

a the speed of the parcel after it has been moving for 2 s **(3 marks)**

b the potential energy lost by the parcel during this time. **(3 marks)**

9 A car of mass 2000 kg is travelling in a straight line at $10\,\text{m}\,\text{s}^{-1}$. The engine of the car produces a constant power of 4000 W. Find the acceleration of the car.

10 A lorry of mass 16 000 kg is travelling up a straight road inclined at 12° to the horizontal. The lorry is travelling at a constant speed of $14\,\text{m}\,\text{s}^{-1}$ and the resistance to motion from non-gravitational forces has a constant magnitude of 200 kN. Find the work done in 10 s by the engine of the lorry.

11 A particle P of mass 0.3 kg is moving in a straight line on a smooth horizontal surface under the action of a constant horizontal force. The particle passes point A with speed $6\,\text{m}\,\text{s}^{-1}$ and point B with speed $12\,\text{m}\,\text{s}^{-1}$.

a Find the kinetic energy gained by P while moving from A to B.

b Write down the work done by the constant force.

The distance from A to B is 4 m.

c Calculate the magnitude of the force.

(E) **12** A box of mass 5 kg slides in a straight line across a rough horizontal floor. The initial speed of the box is $10\,\text{m}\,\text{s}^{-1}$. The only resistance to the motion is the frictional force between the box and the floor. The box comes to rest after moving 8 m. Calculate:

a the kinetic energy lost by the box in coming to rest **(2 marks)**

b the coefficient of friction between the box and the floor. **(4 marks)**

(E) **13** A car of mass 900 kg is moving along a straight horizontal road. The resistance to motion has a constant magnitude. The engine of the car is working at a rate of 15 kW. When the car is moving with speed $20\,\text{m}\,\text{s}^{-1}$, the acceleration of the car is $0.3\,\text{m}\,\text{s}^{-2}$.

a Find the magnitude of the resistance. **(3 marks)**

The car now moves downhill on a straight road inclined at 4° to the horizontal. The engine of the car is now working at a rate of 8 kW. The resistance to motion from non-gravitational forces remains unchanged.

b Calculate the speed of the car when its acceleration is $0.5\,\text{m}\,\text{s}^{-2}$. **(3 marks)**

(E) **14** A bus of mass 7000 kg is travelling in a straight line on a hill inclined at 10° to the horizontal. The engine of the bus produces a constant power of 4000 W and the bus accelerates at $2\,\text{m}\,\text{s}^{-2}$. Find the speed of the bus. **(3 marks)**

(E) **15** A block of wood of mass 4 kg is pulled across a rough horizontal floor by a rope inclined at 15° to the horizontal. The tension in the rope is constant and has magnitude 75 N. The coefficient of friction between the block and the floor is $\frac{3}{8}$

a Find the magnitude of the frictional force opposing the motion. **(4 marks)**

b Find the work done by the tension when the block moves 6 m. **(4 marks)**

The block is initially at rest.

c Find the speed of the block when it has moved 6 m. **(3 marks)**

16 The engine of a lorry works at a constant rate of 20 kW. The lorry has a mass of 1800 kg. When moving along a straight horizontal road there is a constant resistance to motion of magnitude 600 N. Calculate:

a the maximum speed of the lorry

b the acceleration of the lorry, in $\text{m}\,\text{s}^{-2}$, when its speed is $20\,\text{m}\,\text{s}^{-1}$.

(E) **17** A car of mass 1200 kg is travelling at a constant speed of $20\,\text{m}\,\text{s}^{-1}$ along a straight horizontal road. The constant resistance to motion has magnitude 600 N.

a Calculate the power, in kW, developed by the engine of the car. **(4 marks)**

The rate of working of the engine of the car is suddenly increased and the initial acceleration of the car is $0.5\,\text{m}\,\text{s}^{-2}$. The resistance to motion is unchanged.

b Find the new rate of working of the engine of the car. **(5 marks)**

The car now comes to a hill. The road is still straight but is now inclined at 20° to the horizontal. The rate of working of the engine of the car is increased further to 50 kW. The resistance to motion from non-gravitational forces still has magnitude 600 N. The car climbs the hill at a constant speed $v\,\text{m}\,\text{s}^{-1}$.

c Find the value of v. **(5 marks)**

(E/P) **18** An insect of mass 1 g is falling vertically through air. At the instant when its speed is $v\,\text{m}\,\text{s}^{-1}$, the total resistances to motion are modelled as a variable force of magnitude $0.01v^2$ N. Find:

a the acceleration of the insect at the instant when $v = 0.5\,\text{m}\,\text{s}^{-1}$ **(4 marks)**

b the maximum velocity of the insect. **(5 marks)**

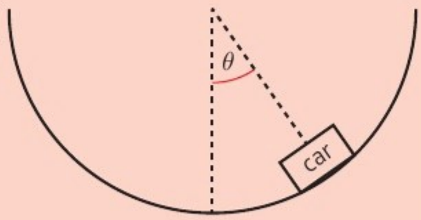
(E/P) **19** A marble of mass 1 kg slides down a plane inclined at 30° to the horizontal. At the instant when its speed is $v\,\text{m}\,\text{s}^{-1}$, the total resistances to motion are modelled as a variable force of magnitude kv N, where k is a constant.

a Find the acceleration of the marble when $v = 1\,\text{m}\,\text{s}^{-1}$, in terms of k. **(4 marks)**

b Given that the maximum velocity of the marble is $5\,\text{m}\,\text{s}^{-1}$, find the value of k. **(2 marks)**

Challenge

A car of mass 3000 kg drives on the inside of a cylindrical tube along a line of steepest slope. It maintains a constant speed of 20 m s^{-1}.



a Find the power generated by the engine when the car is in the position shown in the diagram, giving your answer in terms of θ.

b Explain what happens as $\theta \to 0$ and $\theta \to 90°$.

Summary of key points

1 You can calculate the work done by a force when its point of application moves along a straight line using the formula

work done = component of force in direction of motion × distance moved in direction of force

2 Work done against gravity = mgh, where m is the mass of the particle, g is the acceleration due to gravity and h is the vertical distance raised.

3 Kinetic energy (K.E.) = $\frac{1}{2}mv^2$, where m is the mass of the particle and v is its speed

Potential energy (P.E.) = mgh, where h is the height of the particle above an arbitrary fixed level

4 Work done = change in kinetic energy

5 You must choose a **zero level** of potential energy before calculating a particle's potential energy.

6 **Principle of conservation of mechanical energy**

When no external forces (other than gravity) do work on a particle during its motion, the sum of the particle's kinetic and potential energy remains constant.

7 **Work–energy principle**

The change in the total energy of a particle is equal to the work done on the particle.

8 Power is the rate of doing work.

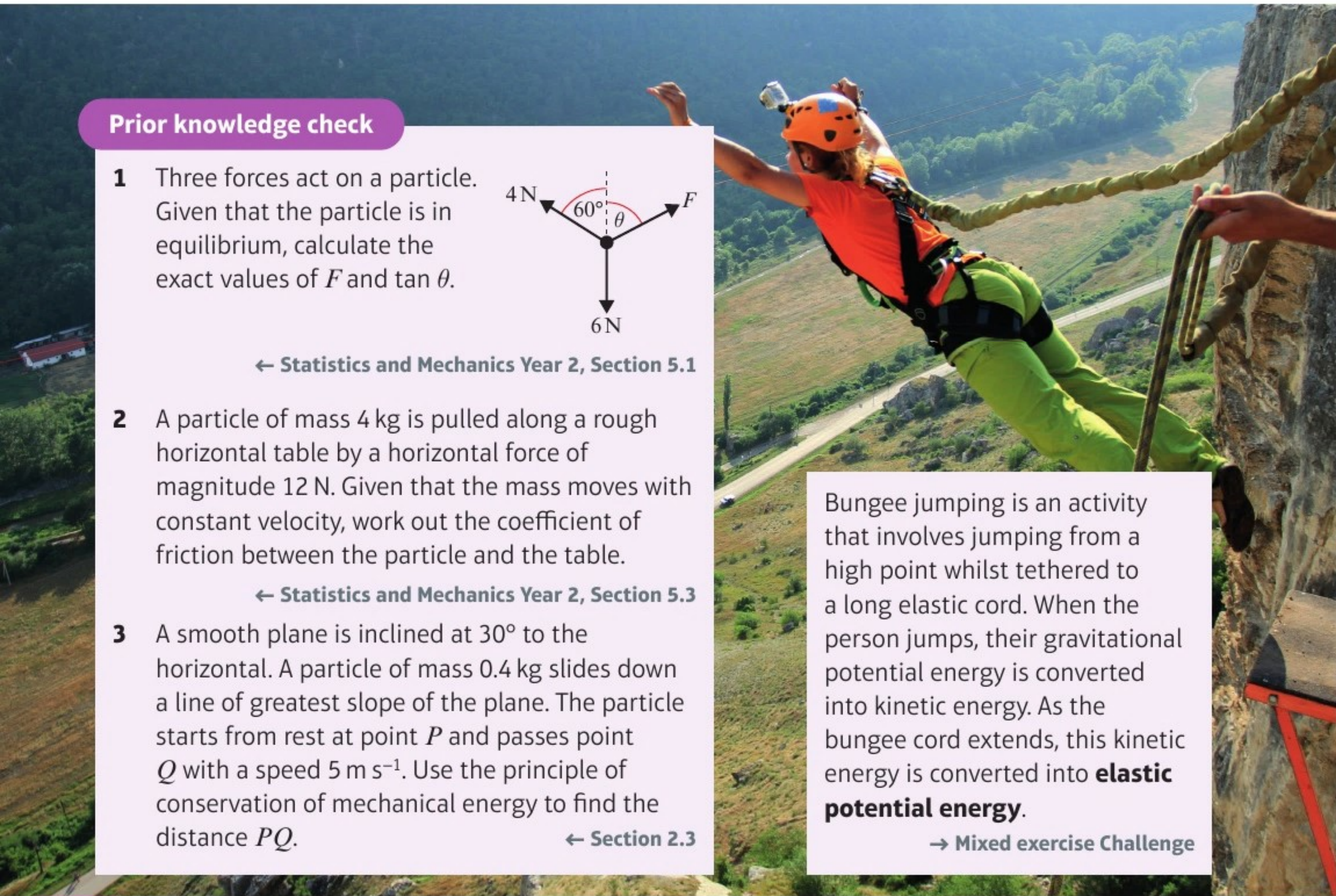
For a vehicle, power = Fv where F is the driving force produced by the engine and v is the speed of the vehicle.

3 Elastic strings and springs

Objectives

After completing this chapter you should be able to:

- Use Hooke's law to solve equilibrium problems involving elastic strings or springs → pages 39–45
- Use Hooke's law to solve dynamics problems involving elastic strings or springs → pages 46–48
- Find the energy stored in an elastic string or spring → pages 49–51
- Solve problems involving elastic energy using the principle of conservation of mechanical energy and the work–energy principle → pages 51–55



Prior knowledge check

1 Three forces act on a particle. Given that the particle is in equilibrium, calculate the exact values of F and $\tan\theta$.

← Statistics and Mechanics Year 2, Section 5.1

2 A particle of mass 4 kg is pulled along a rough horizontal table by a horizontal force of magnitude 12 N. Given that the mass moves with constant velocity, work out the coefficient of friction between the particle and the table.

← Statistics and Mechanics Year 2, Section 5.3

3 A smooth plane is inclined at 30° to the horizontal. A particle of mass 0.4 kg slides down a line of greatest slope of the plane. The particle starts from rest at point P and passes point Q with a speed 5 m s^{-1}. Use the principle of conservation of mechanical energy to find the distance PQ. ← Section 2.3

Bungee jumping is an activity that involves jumping from a high point whilst tethered to a long elastic cord. When the person jumps, their gravitational potential energy is converted into kinetic energy. As the bungee cord extends, this kinetic energy is converted into **elastic potential energy**.

→ Mixed exercise Challenge

3.1 Hooke's law and equilibrium problems

A

You can use Hooke's law to solve equilibrium problems involving elastic strings or springs.

When an elastic string or spring is stretched, the tension, T, produced is proportional to the **extension**, x.

- $T \propto x$
- **$T = kx$, where k is a constant**

The constant k depends on the unstretched length of the string or spring, l, and the **modulus of elasticity** of the string or spring, λ.

- $T = \dfrac{\lambda x}{l}$

This relationship is called **Hooke's law**.

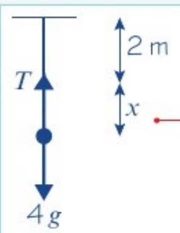
T is a force measured in newtons, and x and l are both lengths, so the units of λ are also newtons. The value of λ depends on the material from which the elastic string or spring is made, and is a measure of the 'stretchiness' of the string or spring. In this chapter you may assume that Hooke's law applies for the values given in a question. In reality, Hooke's law only applies for values of x up to a maximum value, known as the elastic limit of the spring or string.

Watch out An elastic spring can also be **compressed**. Instead of a tension this will produce a **thrust** (or compression) force. Hooke's law still works for compressed elastic springs.

Note In this chapter, all elastic strings and springs are modelled as being **light**. This means they have negligible mass and do not stretch under their own weight.

Example 1

An elastic string of natural length 2 m and modulus of elasticity 29.4 N has one end fixed. A particle of mass 4 kg is attached to the other end and hangs at rest. Find the extension of the string.



Draw a diagram showing all the forces acting on the particle.

Note that the elastic string is in tension.

$(\uparrow)\ T - 4g = 0$

$T = 4g$

The particle is in equilibrium. Resolve vertically upwards to find T.

← Statistics and Mechanics Year 1, Section 10.1

$T = \dfrac{29.4x}{2}$

Using Hooke's law, with $\lambda = 29.4$ and $l = 2$.

so $4g = \dfrac{29.4x}{2}$

Equating the two expressions for T.

$x = \frac{8}{3}$ m

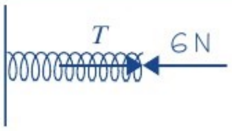
The string stretches by $\frac{8}{3}$ m.

Watch out This is the extension in the spring. The total length will be $4\frac{2}{3}$ m.

Example 2

A

An elastic spring of natural length 1.5 m has one end attached to a fixed point. A horizontal force of magnitude 6 N is applied to the other end and compresses the spring to a length of 1 m. Find the modulus of elasticity of the spring.



Draw a diagram showing the applied force 6 N and the thrust force T produced in the spring.

$(\rightarrow)\ T - 6 = 0$

The forces are in equilibrium.

$T = 6\text{ N}$

$T = \dfrac{\lambda \times 0.5}{1.5}$

Use Hooke's law. The compression of the spring is $1.5 - 1 = 0.5$ m.

$= \dfrac{\lambda}{3}$

so $\dfrac{\lambda}{3} = 6$

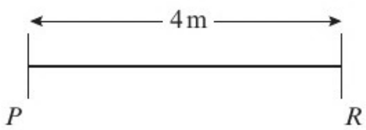
Equating the two expressions for T.

$\lambda = 18\text{ N}$

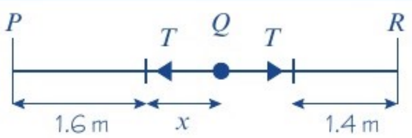
The modulus of elasticity is 18 N.

Example 3

The elastic springs PQ and QR are joined together at Q to form one long spring. The spring PQ has natural length 1.6 m and modulus of elasticity 20 N. The spring QR has natural length 1.4 m and modulus of elasticity 28 N. The ends, P and R, of the long spring are attached to two fixed points which are 4 m apart, as shown in the diagram.



Find the tension in the combined spring.



Problem-solving

Since Q is at rest the tension in each spring must be the same.

Let the extension in spring PQ be x.

The extension in $QR = 1 - x$

Since $PR = 4$ m, total extension of the springs is $4 - 1.6 - 1.4 = 1$ m

For PQ: $T = \dfrac{20x}{1.6}$

For QR: $T = \dfrac{28(1 - x)}{1.4}$

Use Hooke's law.

so $\dfrac{20x}{1.6} = \dfrac{28(1 - x)}{1.4}$

Equate the tensions.

A

$$\frac{20x}{1.6} = 20(1 - x)$$

$$12.5x = 20 - 20x$$

$$32.5x = 20$$

$$x = \frac{8}{13}$$

Solve for x.

so $T = \frac{20}{1.6} \times \frac{8}{13}$

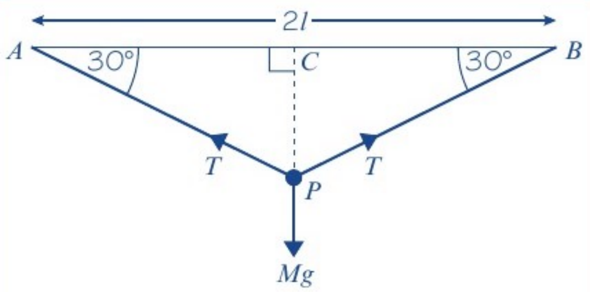
Substitute for x into tension equation for PQ.

$$= \frac{100}{13}\text{ N}$$

The tension in the combined spring is 7.69 N (3 s.f.)

Example 4

An elastic string of natural length $2l$ and modulus of elasticity $4mg$ is stretched between two points A and B. The points A and B are on the same horizontal level and $AB = 2l$. A particle P is attached to the midpoint of the string and hangs in equilibrium with both parts of the string making an angle of 30° with the line AB. Find, in terms of m, the mass of the particle.



Online Explore Hooke's law in equilibrium problems involving two elastic springs using GeoGebra.

Problem-solving

Draw a large clear diagram showing the forces acting on the particle. It is useful to label the midpoint of A and B as well.

Let the mass of the particle be M.

$(\uparrow)\ 2T\cos 60° = Mg$

$T = Mg$

The particle is in equilibrium.

$$AP = \frac{l}{\cos 30°} = \frac{2l}{\sqrt{3}}$$

Use $\triangle APC$.

so the stretched length of the string is

$$\frac{4l}{\sqrt{3}}$$

Since $AP = PB$.

$\therefore$ Extension of string is $\left(\frac{4l}{\sqrt{3}} - 2l\right)$

$$\therefore T = \frac{4mg}{2l}\left(\frac{4l}{\sqrt{3}} - 2l\right)$$

Use Hooke's law.

$$= 2mg\left(\frac{4}{\sqrt{3}} - 2\right)$$

Cancel the ls.

$= 0.62mg$ (2 s.f.)

Hence, $0.62mg = Mg$

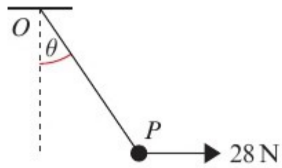
Use $T = mg$.

The mass of the particle is $0.62m$ (2 s.f.).

Example 5

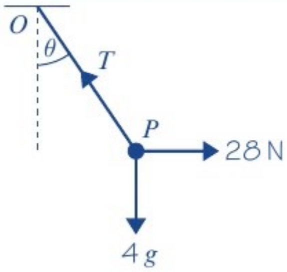
A

An elastic string has natural length 2 m and modulus of elasticity 98 N. One end of the string is attached to a fixed point O and the other end is attached to a particle P of mass 4 kg. The particle is held in equilibrium by a horizontal force of magnitude 28 N, with OP making an angle θ with the vertical, as shown. Find:



a the value of θ

b the length OP.



a $(\leftarrow)\ T\sin\theta = 28$

$(\uparrow)\ T\cos\theta = 4g$

$\tan\theta = \frac{28}{4g} = \frac{5}{7}$

so, $\theta = 35.5°$

$= 36°$ (2 s.f.)

b $T = \frac{28}{\sin\theta}$

so $\frac{28}{\sin\theta} = \frac{98x}{2}$

so $x = \frac{4}{7\sin\theta}$

$= 0.983...$

$OP = 2 + 0.983... = 2.983...$

$\therefore$ Length of OP is 2.98 m (3 s.f.)

Online Explore Hooke's law in equilibrium problems involving one elastic spring using GeoGebra.

Problem-solving

Since the particle is in equilibrium, you can resolve horizontally and vertically to find θ. You could also use these two equations to find an exact value for T, but it is easier to use your calculator and an unrounded value for θ to find x.

Divide the equations to eliminate T.

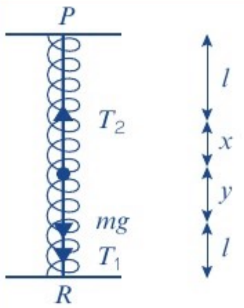
Give answer to 2 s.f. as value for g is correct to 2 s.f.

Use Hooke's law, with x as the extension of the string.

Example 6

Two identical elastic springs PQ and QR each have natural length l and modulus of elasticity $2mg$. The springs are joined together at Q. Their other ends, P and R, are attached to fixed points, with P being $4l$ vertically above R. A particle of mass m is attached at Q and hangs at rest in equilibrium. Find the distance of the particle below P.

A



$l + x + y + l = 4l$ — Since P is $4l$ above R.

$y = 2l - x$

(↑) $T_2 - mg - T_1 = 0$ — Since the mass is in equilibrium.

$\Rightarrow \frac{2mgx}{l} = mg + \frac{2mg(2l - x)}{l}$ — Use Hooke's law.

$2x = l + 2(2l - x)$ — Divide both sides by mg and multiply through by l.

$2x = l + 4l - 2x$

$4x = 5l$

$x = \frac{5l}{4}$ — Solve for x. The extension x is positive and so the top spring is in tension. The value of y is also positive. This demonstrates that the bottom spring is also in tension.

The distance of the particle below P is $\frac{9l}{4}$. — Add on the natural length, l, of the spring.

Problem-solving

Draw a diagram showing the forces acting on the particle. Note that we have assumed that the lower spring is **stretched** and is therefore in **tension**. If the extension of the lower spring turns out to be negative, then it means the lower spring is in compression.

Example 7

One end, A, of a light elastic string AB, of natural length 0.6 m and modulus of elasticity 10 N, is fixed to a point on a fixed rough plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{4}{5}$. A ball of mass 3 kg is attached to the end, B, of the string. The coefficient of friction, μ, between the ball and the plane is $\frac{1}{3}$. The ball rests in limiting equilibrium, on the point of sliding down the plane, with AB along the line of greatest slope.

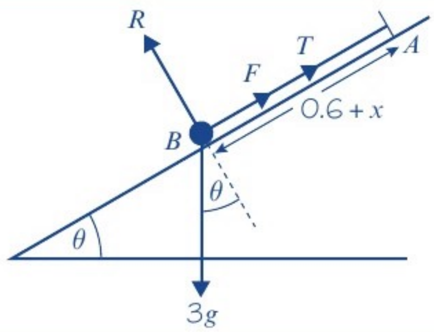
a Find:

i the tension in the string

ii the length of the string.

b If $\mu > \frac{1}{3}$, without doing any further calculation, state how your answer to part **a ii** would change.

a i



Problem-solving

Draw a clear diagram showing all the forces. The ball is on the point of sliding **down** the plane, so the frictional force acts up the plane.

Let extension of string be x m.

$\sin\theta = \frac{4}{5}$ so $\cos\theta = \frac{3}{5}$

($\nwarrow$) $R = 3g\cos\theta = \frac{9g}{5}$

$F = \mu R = \frac{3g}{5}$

($\nearrow$) $T + F = 3g\sin\theta$

$T = 3g\sin\theta - F$

$T = \left(3g \times \frac{4}{5}\right) - \frac{3g}{5} = \frac{9g}{5} = 17.6$ N (3 s.f.)

ii $T = \frac{\lambda x}{l}$

$17.6 = \frac{10x}{0.6}$

so $x = 1.06$ m (3 s.f.)

Length of string $= 0.6 + 1.06$

$= 1.66$ m (3.s.f.)

b If $\mu > \frac{1}{3}$ then

F would be greater as $F = \mu R$

T would be less as $T = 3g\sin\theta - F$

x would be less as $T = \frac{\lambda x}{l}$

so answer to part **a ii** would be less than 1.66 m

Resolving forces perpendicular to the plane.

Ball is in limiting equilibrium so $F = \mu R$.

Resolving forces up the plane and substituting for T and F.

Watch out x is the extension in the string, so add the natural length to find the total length.

Problem-solving

The coefficient of friction is greater, which means that there is a greater force due to friction acting up the plane. The string has to produce less force to keep the ball in equilibrium, so less extension is required.

Exercise 3A

1 One end of a light elastic string is attached to a fixed point. A force of 4 N is applied to the other end of the string so as to stretch it. The natural length of the string is 3 m and the modulus of elasticity is λ N. Find the total length of the string when:

a $\lambda = 30$ **b** $\lambda = 12$ **c** $\lambda = 16$

2 The length of an elastic spring is reduced to 0.8 m when a force of 20 N compresses it. Given that the modulus of elasticity of the spring is 25 N, find its natural length.

(P) 3 An elastic spring of modulus of elasticity 20 N has one end fixed. When a particle of mass 1 kg is attached to the other end and hangs at rest, the total length of the spring is 1.4 m. The particle of mass 1 kg is removed and replaced by a particle of mass 0.8 kg. Find the new length of the spring.

(P) 4 A light elastic spring, of natural length a and modulus of elasticity λ, has one end fixed. A scale pan of mass M is attached to its other end and hangs in equilibrium. A mass m is gently placed in the scale pan. Find the distance of the new equilibrium position below the old one.

A P **5** An elastic string has length a_1 when supporting a mass m_1 and length a_2 when supporting a mass m_2. Find the natural length and modulus of elasticity of the string.

P **6** When a weight, W N, is attached to a light elastic string of natural length l m the extension of the string is 10 cm. When W is increased by 50 N, the extension of the string is increased by 15 cm. Find W.

E/P **7** An elastic spring has natural length $2a$ and modulus of elasticity $2mg$. A particle of mass m is attached to the midpoint of the spring. One end of the spring, A, is attached to the floor of a room of height $5a$ and the other end is attached to the ceiling of the room at a point B vertically above A. The spring is modelled as light.

a Find the distance of the particle below the ceiling when it is in equilibrium. **(8 marks)**

b In reality the spring may not be light. What effect will the model have had on the calculation of the distance of the particle below the ceiling? **(1 mark)**

E/P **8** A uniform rod PQ, of mass 5 kg and length 3 m, has one end, P, smoothly hinged to a fixed point. The other end, Q, is attached to one end of a light elastic string of modulus of elasticity 30 N. The other end of the string is attached to a fixed point R which is on the same horizontal level as P with $RP = 5$ m. The system is in equilibrium and $\angle PQR = 90°$. Find:

a the tension in the string **(5 marks)**

b the natural length of the string. **(3 marks)**

Problem-solving

First take moments about P.

→ **Statistics and Mechanics Year 2, Chapter 4**

E/P **9** A light elastic string AB has natural length l and modulus of elasticity $2mg$. Another light elastic string CD has natural length l and modulus of elasticity $4mg$. The strings are joined at their ends B and C and the end A is attached to a fixed point. A particle of mass m is hung from the end D and is at rest in equilibrium. Find the length AD. **(7 marks)**

E/P **10** An elastic string PA has natural length 0.5 m and modulus of elasticity 9.8 N. The string PB is inextensible. The end A of the elastic string and the end B of the inextensible string are attached to two fixed points which are on the same horizontal level. The end P of each string is attached to a 2 kg particle. The particle hangs in equilibrium below AB, with PA making an angle of 30° with AB and PA perpendicular to PB. Find:

a the length of PA **(7 marks)**

b the length of PB **(2 marks)**

c the tension in PB. **(2 marks)**

E/P **11** A particle of mass 2 kg is attached to one end P of a light elastic string PQ of modulus of elasticity 20 N and natural length 0.8 m. The end Q of the string is attached to a point on a rough plane which is inclined at an angle α to the horizontal, where $\tan \alpha = \frac{3}{4}$. The coefficient of friction between the particle and the plane is $\frac{1}{2}$. The particle rests in limiting equilibrium, on the point of sliding down the plane, with PQ along a line of greatest slope. Find:

a the tension in the string **(5 marks)**

b the length of the string. **(2 marks)**

3.2 Hooke's law and dynamics problems

You can use Hooke's law to solve dynamics problems involving elastic strings or springs.

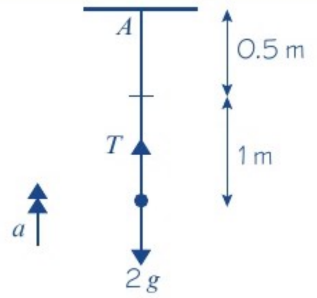
Example 8

One end of a light elastic string, of natural length 0.5 m and modulus of elasticity 20 N, is attached to a fixed point A. The other end of the string is attached to a particle of mass 2 kg. The particle is held at a point which is 1.5 m below A and released from rest. Find:

a the initial acceleration of the particle

b the length of the string when the particle reaches its maximum speed.

a



(↑) $T - 2g = 2a$ — Resolve upwards.

$T = \frac{20 \times 1}{0.5}$ — Use Hooke's law.

$= 40\text{ N}$

so, $40 - 19.6 = 2a$ — Substitute for T.

$10.2 = a$ — Solve for a.

The initial acceleration is 10.2 m s^{-2}

b Particle reaches its maximum speed when it stops accelerating, that is when its acceleration is zero.

$T - 2g = 0$

$T = 2g$ — Maximum speed occurs at the equilibrium position.

$\frac{20x}{0.5} = 2g$

$x = \frac{g}{20}$

$= 0.49$

So the length of the string is

$0.5 + 0.49 = 0.99\text{ m}$ — Add on the natural length to the extension.

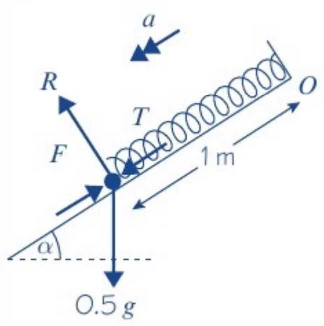
Problem-solving Draw a diagram showing all the forces and the acceleration of the particle. Note that, although the particle is (instantaneously) at rest, it has an upward acceleration.

Watch out Remember that the condition for maximum velocity or speed is $\frac{dv}{dt} = 0$, that is the acceleration = 0. A common misconception is to think the particle reaches maximum speed when the elastic goes slack (i.e. when there is no tension in the spring).

Example 9

A particle of mass 0.5 kg is attached to one end of a light elastic spring of natural length 1.5 m and modulus of elasticity 19.6 N. The other end of the spring is attached to a fixed point O on a rough plane which is inclined to the horizontal at an angle α, where $\tan\alpha = \frac{3}{4}$. The coefficient of friction between the particle and the plane is 0.2. The particle is held at rest on the plane at a point which is 1 m from O down a line of greatest slope of the plane. The particle is released from rest and moves down the slope. Find its initial acceleration.

Online Explore Hooke's law in dynamics problems using GeoGebra.



Problem-solving

Draw a diagram showing all four forces acting on the particle and the acceleration. Note that, since the spring is compressed, it produces a thrust, T, which acts **down** the plane. You can still apply Hooke's law in this situation.

$T = \frac{19.6 \times 0.5}{1.5}$ — By Hooke's law.

$= \frac{19.6}{3}$ N

(↖) $R = 0.5g\cos\alpha$ — There is no acceleration perpendicular to the plane.

$= 4.9 \times \frac{4}{5}$

$= 3.92$ N

so, $F = 0.2 \times 3.92$ — $F = \mu R$ since the particle is about to move.

$= 0.784$ N

(↙) $0.5g\sin\alpha + T - F = 0.5a$ — Resolve down the plane.

$\left(4.9 \times \frac{3}{5}\right) + \frac{19.6}{3} - 0.784 = 0.5a$ — Substitute for T and F.

$2.94 + 6.533 - 0.784 = 0.5a$

$17.37\ldots = a$ — Solve for a.

Initial acceleration is 17 m s^{-2} (2 s.f.)

Exercise 3B

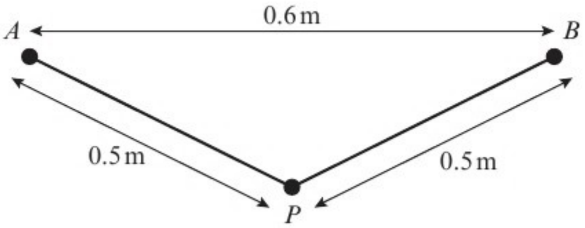
1 A particle of mass 4 kg is attached to one end P of a light elastic spring PQ, of natural length 0.5 m and modulus of elasticity 40 N. The spring rests on a smooth horizontal plane with the end Q fixed. The particle is held at rest and then released. Find the initial acceleration of the particle

a if $PQ = 0.8$ m initially

b if $PQ = 0.4$ m initially.

A E **2** A particle of mass 0.4 kg is fixed to one end A of a light elastic spring AB, of natural length 0.8 m and modulus of elasticity 20 N. The other end B of the spring is attached to a fixed point. The particle hangs in equilibrium. It is then pulled vertically downwards through a distance 0.2 m and released from rest. Find the initial acceleration of the particle. **(4 marks)**

E/P **3**



A particle P of mass 2 kg is attached to the midpoint of a light elastic string, of natural length 0.4 m and modulus of elasticity 20 N. The ends of the elastic string are attached to two fixed points A and B which are on the same horizontal level, with $AB = 0.6$ m. The particle is held in the position shown, with $AP = BP = 0.5$ m, and released from rest. Find the initial acceleration of the particle and state its direction. **(5 marks)**

E/P **4** A particle of mass 2 kg is attached to one end P of a light elastic spring. The other end Q of the spring is attached to a fixed point O. The spring has natural length 1.5 m and modulus of elasticity 40 N. The particle is held at a point which is 1 m vertically above O and released from rest. Find the initial acceleration of the particle, stating its magnitude and direction. **(5 marks)**

E/P **5** A particle of mass 1 kg is attached to one end of a light elastic spring of natural length 1.6 m and modulus of elasticity 21.5 N. The other end of the spring is attached to a fixed point O on a rough plane which is inclined to the horizontal at an angle α where $\tan \alpha = \frac{5}{12}$. The coefficient of friction between the particle and the plane is $\frac{1}{2}$. The particle is held at rest on the plane at a point which is 1.2 m from O down a line of greatest slope of the plane. The particle is released from rest and moves down the slope.

a Find its initial acceleration. **(6 marks)**

b Without doing any further calculation, state how your answer to part **a** would change if the coefficient of friction between the particle and the plane was greater than $\frac{1}{2}$. **(1 mark)**

Challenge

Two light elastic strings each have natural length l and modulus of elasticity λ.
A particle P of mass 3 kg is attached to one end of each string. The other ends of the strings are attached to fixed points A and B, where AB is horizontal and $AB = 2x$ m.
The particle is held at a point x m below the midpoint of AB and released from rest.
The initial acceleration of the particle is $\frac{g}{2}$ m s^{-2}.

a Show that when the particle is released the tension, T, in each string is $\frac{3\sqrt{2}g}{4}$ N.

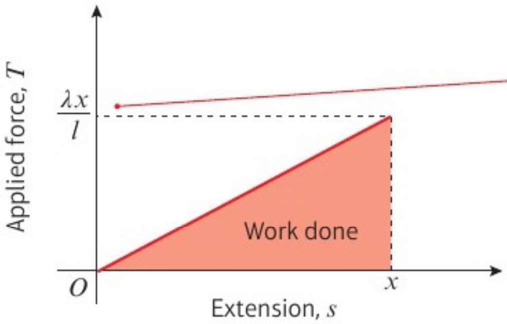
b Given that at the point the particle is released, each string has extended by $\frac{1}{4}$ of its natural length, find the modulus of elasticity for each string.

3.3 Elastic energy

A

You can find the energy stored in an elastic string or spring.

You can draw a **force–distance** diagram to show the extension x in an elastic string as a gradually increasing force is applied. The area under the force–distance graph is the **work done** in stretching the elastic string.



The applied force is always equal and opposite to the tension in the elastic string, T. This value increases as the string stretches.

Using the formula for the area of a triangle:

$$\text{Area} = \tfrac{1}{2}x\left(\frac{\lambda x}{l}\right) = \frac{\lambda x^2}{2l}$$

Using integration:

$$\text{Area} = \int_0^x T\,\mathrm{d}s = \int_0^x \frac{\lambda s}{l}\,\mathrm{d}s = \left[\frac{\lambda s^2}{2l}\right]_0^x = \frac{\lambda x^2}{2l}$$

- **The work done in stretching an elastic string or spring of modulus of elasticity λ from its natural length l to a length $(l + x)$ is $\frac{\lambda x^2}{2l}$.**

Watch out This rule applies as long as the extension is within the elastic limit of the string or spring.

When λ is measured in newtons and x and l are measured in metres, the work done is in **joules (J)**.

When a stretched string is released it will 'ping' back to its natural length. In its stretched position it has the potential to do work, or **elastic potential energy** (this is also called **elastic energy**).

- **The elastic potential energy (E.P.E.) stored in a stretched string or spring is exactly equal to the amount of work done to stretch the string or spring.**

Links This is an application of the work–energy principle. ← Section 2.3

- **The E.P.E. stored in an elastic string or spring of modulus of elasticity λ which has been stretched from its natural length l to a length $(l + x)$ is $\frac{\lambda x^2}{2l}$.**

You can apply the same formulae for work done and elastic potential energy when an elastic string or spring is compressed.

Example 10

A

An elastic string has natural length 1.4 m and modulus of elasticity 6 N. Find the energy stored in the string when its length is 1.6 m.

Energy stored $= \frac{6 \times 0.2^2}{2 \times 1.4}$
$= 0.0857$ J (3 s.f.)

Use $\frac{\lambda x^2}{2l}$ with $x = 1.6 - 1.4 = 0.2$

Example 11

A light elastic spring has natural length 0.6 m and modulus of elasticity 10 N. Find the work done in compressing the spring from a length of 0.5 m to a length of 0.3 m.

Work done in compression = Energy stored when length is 0.3 m − Energy stored when length is 0.5 m

$= \frac{10 \times 0.3^2}{2 \times 0.6} - \frac{10 \times 0.1^2}{2 \times 0.6}$

$= \frac{10}{1.2}(0.3^2 - 0.1^2)$

$= \frac{25}{3}(0.3 + 0.1)(0.3 - 0.1)$

$= \frac{25}{3} \times 0.4 \times 0.2$

$= \frac{2}{3}$ J

The spring is being compressed, so it will have greater stored energy at the shorter length.

Use the **compression** in the formula, not the length.

Watch out A common error is to use: $\frac{10 \times (0.3 - 0.1)^2}{2 \times 0.6}$ which is not the same.

Exercise 3C

1 An elastic spring has natural length 0.6 m and modulus of elasticity 8 N. Find the work done when the spring is stretched from its natural length to a length of 1 m.

2 An elastic spring, of natural length 0.8 m and modulus of elasticity 4 N, is compressed to a length of 0.6 m. Find the elastic potential energy stored in the spring.

3 An elastic string has natural length 1.2 m and modulus of elasticity 10 N. Find the work done when the string is stretched from a length of 1.5 m to a length 1.8 m.

4 An elastic spring has natural length 0.7 m and modulus of elasticity 20 N. Find the work done when the spring is stretched from a length

a 0.7 m to 0.9 m

b 0.8 m to 1.0 m

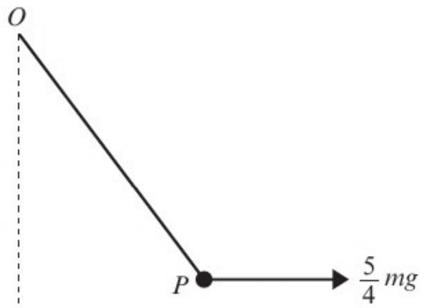
c 1.2 m to 1.4 m

Hint Note that your answers to **a**, **b** and **c** are all different.

A E **5** A light elastic spring has natural length 1.2 m and modulus of elasticity 10 N. One end of the spring is attached to a fixed point. A particle of mass 2 kg is attached to the other end and hangs in equilibrium. Find the energy stored in the spring. **(3 marks)**

E/P **6** An elastic string has natural length a. One end is fixed. A particle of mass $2m$ is attached to the free end and hangs in equilibrium, with the length of the string $3a$. Find the elastic potential energy stored in the string. **(3 marks)**

E/P **7**



A particle P of mass m is attached to one end of a light elastic string, of natural length a and modulus of elasticity $2mg$. The other end of the string is attached to a fixed point O.

The particle P is held in equilibrium by a horizontal force of magnitude $\frac{5}{4}mg$ applied to P.

This force acts in the vertical plane containing the string, as shown in the diagram.

Find:

a the tension in the string **(5 marks)**

b the elastic energy stored in the string. **(4 marks)**

3.4 Problems involving elastic energy

You can solve problems involving elastic energy using the principle of conservation of mechanical energy and the work–energy principle.

- **When no external forces (other than gravity) act on a particle, then the sum of its kinetic energy, gravitational potential energy and elastic potential energy remains constant.**

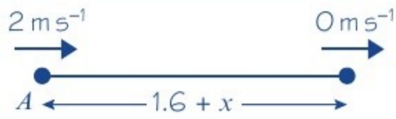
Links This is an application of the principle of conservation of mechanical energy. ← **Section 2.3**

If a particle which is attached to an elastic spring or string is subject to a resistance as it moves, you will need to apply the work–energy principle.

Example 12

A light elastic string, of natural length 1.6 m and modulus of elasticity 10 N, has one end fixed at a point A on a smooth horizontal table. A particle of mass 2 kg is attached to the other end of the string. The particle is held at the point A and projected horizontally along the table with speed 2 m s^{-1}. Find how far it travels before first coming to instantaneous rest.

A



Draw a simple diagram showing the initial and final positions of the particle.

Suppose that the extension of the string when the particle comes to rest is x.

K.E. lost by the particle = E.P.E. gained by the string

You can apply the conservation of energy principle since the table is smooth.

$$\tfrac{1}{2}mv^2 = \frac{\lambda x^2}{2l}$$

$$\tfrac{1}{2} \times 2 \times 2^2 = \frac{10x^2}{2 \times 1.6}$$

Note that you do not need to consider the energy of the particle in any intermediate positions.

$$1.28 = x^2$$

$$1.13\ldots = x$$

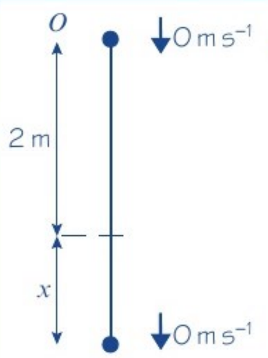
Total distance travelled is 2.73 m (3 s.f.)

Add on the natural length of the string.

It is important to realise that in the example above, the particle is not in equilibrium when it comes to instantaneous rest and so you cannot use forces to solve this type of problem. The particle in fact has an acceleration and will immediately 'spring' back towards A.

Example 13

A particle of mass 0.5 kg is attached to one end of an elastic string, of natural length 2 m and modulus of elasticity 19.6 N. The other end of the elastic string is attached to a point O. The particle is released from the point O. Find the greatest distance it will reach below O.



Draw a diagram showing the initial and final positions of the particle. Let the extension of the string be x when the particle comes to rest.

P.E. lost by particle = E.P.E. gained by string

There is no K.E. involved as the particle starts at rest and finishes at rest. Assuming no air resistance, energy will be conserved.

$$mgh = \frac{\lambda x^2}{2l}$$

$$0.5g(2 + x) = \frac{19.6x^2}{4}$$

$$2 + x = x^2$$

$$0 = x^2 - x - 2$$

$$0 = (x - 2)(x + 1)$$

$$x = 2 \text{ or } -1$$

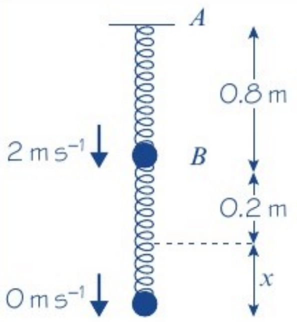
Hence greatest distance reached below O is 4 m.

Problem-solving

x is the extension in the string so it must be positive. Ignore the negative solution, and remember to add the natural length of the string.

Example 14

A light elastic spring, of natural length 1 m and modulus of elasticity 10 N, has one end attached to a fixed point A. A particle of mass 2 kg is attached to the other end of the spring and is held at a point B which is 0.8 m vertically below A. The particle is projected vertically downwards from B with speed $2\,\text{m s}^{-1}$. Find the distance it travels before first coming to rest.



Let the extension of the spring be x when the particle comes to rest.

K.E. loss + P.E. loss = E.P.E. gain

$$\frac{1}{2} \times 2 \times 2^2 + 2g(0.2 + x) = \frac{10x^2}{2} - \frac{10 \times 0.2^2}{2}$$

$$4 + 3.92 + 19.6x = 5x^2 - 0.2$$

$$0 = 5x^2 - 19.6x - 8.12$$

$$x = \frac{19.6 \pm \sqrt{19.6^2 + (4 \times 5 \times 8.12)}}{10}$$

$$= \frac{19.6 \pm 23.37\ldots}{10}$$

$x = 4.298\ldots$ or $-0.378\ldots$

Distance travelled $= 4.298\ldots + 0.2$

$= 4.498\ldots$

$= 4.5$ m (2 s.f.)

Problem-solving

You will need to use the principle of conservation of mechanical energy with kinetic energy, gravitational potential energy **and** elastic potential energy.

Let the extension of the spring be x.
E.P.E. gain = final E.P.E. – initial E.P.E.

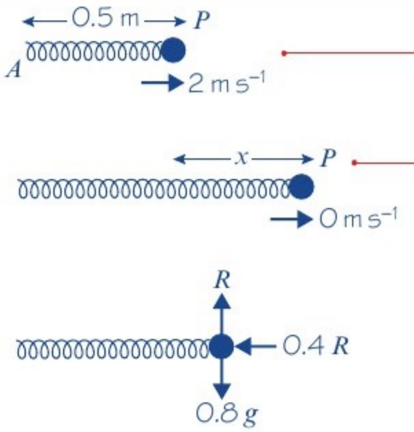
Write in the form $ax^2 + bx + c = 0$ to solve quadratic.

Ignore the negative solution.

The spring is compressed by 0.2 m at the start.

Example 15

A light elastic spring, of natural length 0.5 m and modulus of elasticity 10 N, has one end attached to a point A on a rough horizontal plane. The other end is attached to a particle P of mass 0.8 kg. The coefficient of friction between the particle and the plane is 0.4. The particle initially lies on the plane with $AP = 0.5$ m and is then projected with speed $2\,\text{m s}^{-1}$ away from A along the plane. Find the distance moved by P before it first comes to rest.



Draw a diagram showing the initial and final positions of the particle.

Let the extension of the spring be x.

Problem-solving

As it slides, P will be moving against friction, μR, from the plane.

($\uparrow$) $R = 0.8g$

First find the magnitude of the friction force.

Friction $= 0.4 \times 0.8g$
$= 0.32g$

work done against friction = overall loss in energy

Apply the work–energy principle.

work done against friction = K.E. loss of P − E.P.E. gain of spring

$$0.32gx = \frac{1}{2} \times 0.8 \times 2^2 - \frac{10x^2}{2 \times 0.5}$$

Use force × distance, $\frac{1}{2}mv^2$ and $\frac{\lambda x^2}{2l}$

$$10x^2 + 0.32gx - 1.6 = 0$$

Write in the form $ax^2 + bx + c = 0$ to solve quadratic.

$$x = \frac{-0.32g \pm \sqrt{(0.32g)^2 + 64}}{20}$$

Solve for x.

$x = 0.2728\ldots$ or $-0.586\ldots$

Ignore the negative solution.

P moves a distance 0.27 m (2 s.f.) before coming to rest.

Exercise 3D

P **1** An elastic string, of natural length l and modulus of elasticity mg, has one end fixed to a point A on a smooth horizontal table. The other end is attached to a particle P of mass m. The particle is held at a point on the table with $AP = \frac{3}{2}l$ and is released. Find the speed of the particle when the string reaches its natural length.

P **2** A particle of mass m is suspended from a fixed point O by a light elastic string, of natural length a and modulus of elasticity $4mg$. The particle is pulled vertically downwards a distance d from its equilibrium position and released from rest. If the particle just reaches O, find d.

E/P **3** A light elastic spring of natural length $2l$ has its ends attached to two points P and Q which are at the same horizontal level. The length PQ is $2l$. A particle of mass m is fastened to the midpoint of the spring and is held at the midpoint of PQ. The particle is released from rest and first comes to instantaneous rest when both parts of the spring make an angle of 60° with the line PQ.

A

a Find the modulus of elasticity of the spring. **(6 marks)**

b Suggest one way in which the model could be refined to make it more realistic. **(1 mark)**

E/P **4** A light elastic string, of natural length 1 m and modulus of elasticity 21.6 N, has one end attached to a fixed point O. A particle of mass 2 kg is attached to the other end. The particle is held at a point which is 3 m vertically below O and released from rest. Find:

a the speed of the particle when the string first becomes slack **(5 marks)**

b the distance from O when the particle first comes to rest. **(3 marks)**

E/P **5** A particle P is attached to one end of a light elastic string of natural length a. The other end of the string is attached to a fixed point O. When P hangs at rest in equilibrium, the distance OP is $\frac{5a}{3}$. The particle is now projected vertically downwards from O with speed U and first comes to instantaneous rest at a distance $\frac{10a}{3}$ below O. Find U in terms of a and g. **(7 marks)**

E/P **6** A particle P of mass 1 kg is attached to the midpoint of a light elastic string, of natural length 3 m and modulus λ N. The ends of the string are attached to two points A and B on the same horizontal level with $AB = 3$ m. The particle is held at the midpoint of AB and released from rest. The particle falls vertically and comes to instantaneous rest at a point which is 1 m below the midpoint of AB. Find:

a the value of λ **(5 marks)**

b the speed of P when it is 0.5 m below the initial position. **(5 marks)**

E/P **7** A light elastic string of natural length 2 m and modulus of elasticity 117.6 N has one end attached to a fixed point O. A particle P of mass 3 kg is attached to the other end. The particle is held at O and released from rest.

a Find the distance fallen by P before it first comes to rest. **(5 marks)**

b Find the greatest speed of P during the fall. **(4 marks)**

Problem-solving

P will be travelling at its greatest speed when the acceleration = 0.

E/P **8** A particle P of mass 2 kg is attached to one end of a light elastic string of natural length 1 m and modulus of elasticity 40 N. The other end of the string is fixed to a point O on a rough plane which is inclined at an angle α, where $\tan \alpha = \frac{3}{4}$. The particle is held at O and released from rest. Given that P comes to rest after moving 2 m down the plane, find the coefficient of friction between the particle and the plane. **(4 marks)**

Challenge

An elastic string of natural length l m is suspended from a fixed point O. When a mass of M kg is attached to the other end of the string, its extension is $\frac{l}{10}$ m. An additional M kg is then attached to the end of the string. Show that the work done in producing the additional extension is $\frac{3Mgl}{20}$ J.

Mixed exercise 3

A

P **1** A particle of mass m is supported by two light elastic strings, each of natural length a and modulus of elasticity $\frac{15mg}{16}$. The other ends of the strings are attached to two fixed points A and B where A and B are in the same horizontal line with $AB = 2a$. When the particle hangs at rest in equilibrium below AB, each string makes an angle θ with the vertical.

a Verify that $\cos\theta = \frac{4}{5}$.

b How much work must be done to raise the particle to the midpoint of AB?

2 A light elastic spring is such that a weight of magnitude W resting on the spring produces a compression a. The weight W is allowed to fall onto the spring from a height of $\frac{3a}{2}$ above it. Find the maximum compression of the spring in the subsequent motion.

3 A light elastic string of natural length 0.5 m is stretched between two points P and Q on a smooth horizontal table. The distance PQ is 0.75 m and the tension in the string is 15 N.

a Find the modulus of elasticity of the string.

A particle of mass 0.5 kg is attached to the midpoint of the string. The particle is pulled 0.1 m towards Q and released from rest.

b Find the speed of the particle as it passes through the midpoint of PQ.

P **4** A particle of mass m is attached to two strings AP and BP. The points A and B are on the same horizontal level and $AB = \frac{5a}{4}$.

The string AP is inextensible and $AP = \frac{3a}{4}$.

The string BP is elastic and $BP = a$.

The modulus of elasticity of BP is λ. Show that the natural length of BP is $\frac{5\lambda a}{3mg + 5\lambda}$

P **5** A light elastic string, of natural length a and modulus of elasticity $5mg$, has one end attached to the base of a vertical wall. The other end of the string is attached to a small ball of mass m. The ball is held at a distance $\frac{3a}{2}$ from the wall, on a rough horizontal plane, and released from rest. The coefficient of friction between the ball and the plane is $\frac{1}{5}$.

a Find, in terms of a and g, the speed V of the ball as it hits the wall.

The ball rebounds from the wall with speed $\frac{2V}{5}$. The string stays slack.

b Find the distance from the wall at which the ball comes to rest.

A E/P **6** A light elastic string has natural length l and modulus $2mg$. One end of the string is attached to a particle P of mass m. The other end is attached to a fixed point C on a rough horizontal plane. Initially P is at rest at a point D on the plane where $CD = \frac{4l}{3}$.

a Given that P is in limiting equilibrium, find the coefficient of friction between P and the plane. **(5 marks)**

The particle P is now moved away from C to a point E on the plane where $CE = 2l$.

b Find the speed of P when the string returns to its natural length. **(5 marks)**

c Find the total distance moved by P before it comes to rest. **(4 marks)**

E/P **7** A light elastic string of natural length 0.2 m has its ends attached to two fixed points A and B which are on the same horizontal level with $AB = 0.2$ m. A particle of mass 5 kg is attached to the string at the point P where $AP = 0.15$ m. The system is released and P hangs in equilibrium below AB with $\angle APB = 90°$.

a If $\angle BAP = \theta$, show that the ratio of the extension of AP and BP is

$$\frac{4\cos\theta - 3}{4\sin\theta - 1}$$ **(4 marks)**

b Hence show that

$$\cos\theta\,(4\cos\theta - 3) = 3\sin\theta\,(4\sin\theta - 1).$$ **(4 marks)**

E/P **8** A particle of mass 3 kg is attached to one end of a light elastic string, of natural length 1 m and modulus of elasticity 14.7 N. The other end of the string is attached to a fixed point. The particle is held in equilibrium by a horizontal force of magnitude 9.8 N with the string inclined to the vertical at an angle θ.

a Find the value of θ. **(3 marks)**

b Find the extension of the string. **(3 marks)**

c If the horizontal force is removed, find the magnitude of the least force that will keep the string inclined at the same angle. **(2 marks)**

E/P **9** Two points A and B are on the same horizontal level with $AB = 3a$. A particle P of mass m is joined to A by a light inextensible string of length $4a$ and is joined to B by a light elastic string, of natural length a and modulus of elasticity $\frac{mg}{4}$. The particle P is held at a point C, such that $BC = a$ and both strings are taut. The particle P is released from rest.

a Show that when AP is vertical the speed of P is $2\sqrt{ga}$. **(6 marks)**

b Find the tension in the elastic string in this position. **(4 marks)**

Challenge

A A bungee jumper attaches one end of an elastic rope to both ankles. The other end is attached to the platform on which he stands.

The bungee jumper is modelled as a particle of mass m kg attached to an elastic string of natural length l m with modulus of elasticity λ N.

a Show that the maximum distance the jumper descends after jumping off the platform is $l + k + \sqrt{2kl + k^2}$ where $k = \frac{mgl}{\lambda}$.

b Suggest a refinement to this model that would result in:

i a greater maximum descent

ii a smaller maximum descent.

Summary of key points

1 When an elastic string or spring is stretched, the tension, T, produced is proportional to the extension, x.

- $\boldsymbol{T \propto x}$
- $\boldsymbol{T = kx}$**, where k is a constant**

The constant k depends on the unstretched length of the spring or string, l, and the **modulus of elasticity** of the string or spring, λ.

- $\boldsymbol{T = \frac{\lambda x}{l}}$

This relationship is called **Hooke's law**.

2 The area under a **force–distance** graph is the **work done** in stretching an elastic string or spring. The work done in stretching or compressing an elastic string or spring with modulus of elasticity λ from its natural length, l to a length $(l + x)$ is $\frac{\lambda x^2}{2l}$.

When λ is measured in newtons and x and l are measured in metres, the work done is in **joules (J)**.

3 The **elastic potential energy** (E.P.E.) stored in a stretched string or spring is exactly equal to the amount of work done to stretch the string or spring.

The E.P.E. stored in a string or spring of modulus of elasticity λ which has been stretched from its natural length l, to a length $(l + x)$ is $\frac{\lambda x^2}{2l}$.

4 When no external forces (other than gravity) act on a particle, then the sum of its kinetic energy, gravitational potential energy and elastic potential energy remains constant.

Review exercise 1

E/P **1** A ball of mass 0.3 kg is released at rest from a point at a height of 10 m above horizontal ground. After hitting the ground the ball rebounds to a height of 2.5 m.

Calculate the magnitude of the impulse exerted by the ground on the ball. **(4)**

← Section 1.1

E/P **2** A toy racing car of mass 250 g is travelling on a smooth horizontal surface with momentum 2 N s. At point A it passes onto a rough horizontal surface where the coefficient of friction $\mu = 0.2$.

a Modelling the car as a particle, find, to the nearest metre, the distance taken for the racing car to come to a complete stop. **(6)**

b In reality the car stops in a shorter distance. Suggest one reason for this. **(1)**

← Section 1.1

E **3** Two particles A and B have masses 0.4 kg and 0.3 kg respectively. They are moving in opposite directions on a smooth horizontal table and collide directly. Immediately before the collision, the speed of A is $6\,\text{m s}^{-1}$ and the speed of B is $2\,\text{m s}^{-1}$. As a result of the collision, the direction of motion of B is reversed and its speed immediately after the collision is $3\,\text{m s}^{-1}$. Find:

a the speed of A immediately after the collision, stating clearly whether the direction of motion of A is changed by the collision **(3)**

b the magnitude of the impulse exerted on B in the collision, stating clearly the units in which your answer is given. **(3)**

← Sections 1.1, 1.2

E/P **4** A railway truck S of mass 2000 kg is travelling due east along a straight horizontal track with constant speed $12\,\text{m s}^{-1}$. The truck S collides with a truck T which is travelling due west along the same track as S with constant speed $6\,\text{m s}^{-1}$. The magnitude of the impulse of T on S is 28 800 N s.

a Calculate the speed of S immediately after the collision. **(2)**

b State whether or not the motion of S is changed by the collision. **(1)**

Given that, immediately after the collision, the speed of T is $3.6\,\text{m s}^{-1}$, and that T and S are moving in opposite directions,

c calculate the mass of T. **(3)**

← Sections 1.1, 1.2

E/P **5** Two particles A and B, of masses 0.5 kg and 0.4 kg respectively, are travelling in the same straight line on a smooth horizontal table. Particle A, moving with speed $3\,\text{m s}^{-1}$, strikes particle B, which is moving with speed $2\,\text{m s}^{-1}$ in the same direction. After the collision A and B are moving in the same direction and the speed of B is $0.8\,\text{m s}^{-1}$ greater than the speed of A.

a Find the speed of A and the speed of B after the collision. **(5)**

b Show that A loses momentum 0.4 N s in the collision. **(3)**

Particle B later hits an obstacle on the table and rebounds in the opposite direction with speed $1\,\text{m}\,\text{s}^{-1}$.

c Find the magnitude of the impulse received by B in this second impact. **(3)**

← Sections 1.1, 1.2

E/P **6** Two particles A and B, of masses 3 kg and 2 kg respectively, are moving in the same direction on a smooth horizontal table when they collide directly. Immediately before the collision, the speed of A is $4\,\text{m}\,\text{s}^{-1}$ and the speed of B is $1.5\,\text{m}\,\text{s}^{-1}$. In the collision, the particles join to form a single particle C.

Find the speed of C immediately after the collision. **(3)**

← Sections 1.1, 1.2

E/P **7** Two particles P and Q have masses 3 kg and m kg respectively. They are moving towards each other in opposite directions on a smooth horizontal table. Each particle has speed $4\,\text{m}\,\text{s}^{-1}$, when they collide directly. In this collision, the direction of motion of each particle is reversed. The speed of P immediately after the collision is $2\,\text{m}\,\text{s}^{-1}$ and the speed of Q is $1\,\text{m}\,\text{s}^{-1}$.

Find:

a the value of m **(3)**

b the magnitude of the impulse exerted on Q in the collision. **(2)**

← Sections 1.1, 1.2

A E **8** In this question **i** and **j** are perpendicular unit vectors in a horizontal plane.

A ball has a mass 0.2 kg. It is moving with velocity $30\mathbf{i}\,\text{m}\,\text{s}^{-1}$ when it is struck by a bat. The bat exerts an impulse of $(-4\mathbf{i} + 4\mathbf{j})$ Ns on the ball. Find:

a the velocity of the ball immediately after the impact **(3)**

b the angle through which the ball is deflected as a result of the impact **(2)**

c the kinetic energy lost by the ball in the impact. **(4)**

← Sections 1.3, 2.2

E/P **9** A particle P of mass 0.75 kg is moving under the action of a single force **F** newtons. At time t seconds, the velocity $\mathbf{v}\,\text{m}\,\text{s}^{-1}$ of P is given by

$$\mathbf{v} = (t^2 + 2)\mathbf{i} - 6t\mathbf{j}$$

a Find the magnitude of **F** when $t = 4$. **(5)**

When $t = 5$, the particle P receives an impulse of magnitude $9\sqrt{2}$ Ns in the direction of the vector $\mathbf{i} - \mathbf{j}$.

b Find the velocity of P immediately after the impulse. **(4)**

← Section 1.3

E **10** A tennis ball of mass 0.2 kg is moving with velocity $-10\mathbf{i}\,\text{m}\,\text{s}^{-1}$ when it is struck by a tennis racket. Immediately after being struck, the ball has velocity $(15\mathbf{i} + 15\mathbf{j})\,\text{m}\,\text{s}^{-1}$. Find:

a the magnitude of the impulse exerted by the racket on the ball **(4)**

b the angle, to the nearest degree, between the vector **i** and the impulse exerted by the racket **(2)**

c the kinetic energy gained by the ball as a result of being struck. **(2)**

← Sections 1.3, 2.2

E **11** At time t seconds the acceleration, $\mathbf{a}\,\text{m}\,\text{s}^{-2}$, of a particle P relative to a fixed origin O, is given by $\mathbf{a} = 2\mathbf{i} + t\mathbf{j}$. Initially the velocity of P is $(2\mathbf{i} - 4\mathbf{j})\,\text{m}\,\text{s}^{-1}$.

a Find the velocity of P at time t seconds. **(3)**

At time $t = 2$ seconds the particle P is given an impulse $(3\mathbf{i} - 1.5\mathbf{j})$ Ns.

Given that the particle P has mass 0.5 kg,

b find the speed of P immediately after the impulse has been applied. **(6)**

← Section 1.3

(E) **12** A rough plane is inclined at an angle of 5° to the horizontal. A sled of mass 1250 kg is pushed up a line of greatest slope of the plane. The coefficient of friction between the sled and the plane is 0.05.

a Find the magnitude of the frictional force acting on the sled. **(5)**

The sled moves a total distance of 750 m, as measured along the line of greatest slope of the plane. Find:

b the work done against friction **(3)**

c the work done against gravity. **(4)**

← Section 2.1

(E/P) **13** A winch pulls a sled of mass 1000 kg, a distance of 25 m up the line of greatest slope of a ramp. Given that the work done against gravity by the winch is 19.6 kJ, show that the slope is inclined at an angle of $\arcsin\left(\frac{2}{25}\right)$ to the horizontal. **(5)**

← Section 2.1

(E) **14** A rock of mass 4 kg is dropped from rest at the top of a cliff. It falls 40 m vertically down before hitting the surface of the sea.

a Modelling the rock as a particle falling freely under gravity, find its kinetic energy when it hits the surface of the sea. **(2)**

In reality the falling rock will be subject to air resistance which will oppose its motion.

b Without further calculation, state how this will affect the kinetic energy of the rock when it hits the surface of the water. **(1)**

← Section 2.2

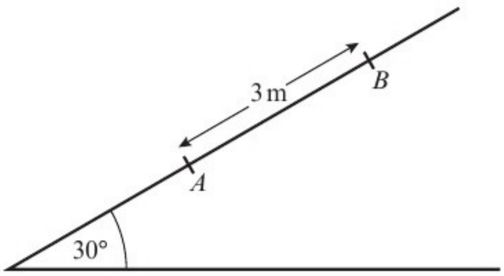
(E) **15** A cable car of mass 200 kg moves along a section of cable which can be modelled as a straight line inclined at 30° above the horizontal. As the cable car moves up the cable for 200 m its speed reduces from 2 m s^{-1} to 1.5 m s^{-1}. Calculate:

a the loss in kinetic energy of the cable car **(4)**

b the gain in potential energy of the cable car. **(4)**

← Section 2.2

(E) **16**



A particle P of mass 2 kg is projected from a point A up a line of greatest slope AB of a fixed plane. The plane is inclined at an angle of 30° to the horizontal and $AB = 3$ m with B above A, as shown in the figure. The speed of P at A is 10 m s^{-1}.

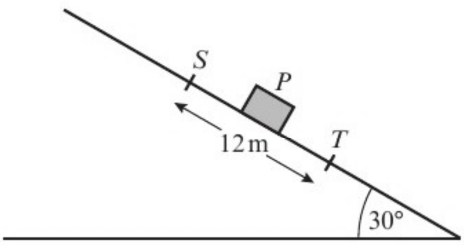
a Assuming the plane is smooth, find the speed of P at B. **(4)**

The plane is now assumed to be rough. At A the speed of P is 10 m s^{-1} and at B the speed of P is 7 m s^{-1}.

b By using the work–energy principle, or otherwise, find the coefficient of friction between P and the plane. **(5)**

← Section 2.3

(E) **17**



A small package is modelled as a particle P of mass 0.6 kg. The package slides down a rough plane from a point S to a point T, where $ST = 12$ m. The plane is inclined at 30° to the horizontal and ST is a line of greatest slope of the plane, as shown in the figure. The speed of P at S is 10 m s^{-1} and the speed of P at T is 9 m s^{-1}. Calculate:

a the total loss of energy of P in moving from S to T **(4)**

b the coefficient of friction between P and the plane. **(5)**

← Section 2.3

E/P **18** A particle P has mass 4 kg. It is projected from a point A up a line of greatest slope of a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The coefficient of friction between P and the plane is $\frac{2}{7}$. The particle comes to rest instantaneously at the point B on the plane, where $AB = 2.5$ m. It then moves back down the plane to A.

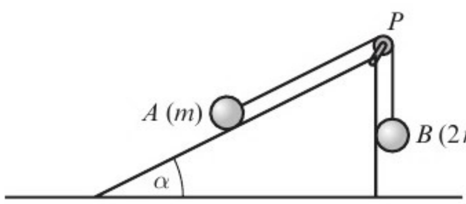
a Find the work done by friction as P moves from A to B. **(4)**

b Using the work–energy principle, find the speed with which P is projected from A. **(4)**

c Find the speed of P when it returns to A. **(4)**

← Section 2.3

E/P **19**



Two particles A and B of masses m and $2m$ respectively are attached to the ends of a light inextensible string. The particle A lies on a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The string passes over a small light pulley P fixed at the top of the plane. The particle B hangs freely below P, as shown in the figure. The particles are released from rest with the string taut and the section of the string from A to P parallel to a line of greatest slope of the plane. The coefficient of friction between A and the plane is $\frac{5}{8}$. When each particle has moved a distance h, B has not reached the ground and A has not reached P.

a Find an expression for the potential energy lost by the system when each particle has moved a distance h. **(2)**

When each particle has moved a distance h, they are moving with speed v.

b Using the work–energy principle, find an expression for v^2, giving your answer in the form kgh where k is a number. **(5)**

← Section 2.3

E **20** A car of mass 1000 kg is towing a trailer of mass 1500 kg along a straight horizontal road. The tow-bar joining the car to the trailer is modelled as a light rod parallel to the road. The total resistance to motion of the car is modelled as having constant magnitude 750 N. The total resistance to motion of the trailer is modelled as a force of magnitude R newtons, where R is a constant. When the engine is working at a rate of 50 kW, the car and the trailer travel at a constant speed of 25 m s^{-1}.

a Show that $R = 1250$. **(3)**

When travelling at 25 m s^{-1} the driver of the car disengages the engine and applies the brakes. The brakes provide a constant braking force of magnitude 1500 N to the car. The resisting forces of magnitude 750 N and 1250 N are assumed to remain unchanged. Calculate:

b the deceleration of the car while braking **(3)**

c the thrust in the tow-bar while braking **(2)**

d the work done, in kJ, by the braking force in bringing the car and the trailer to rest. **(4)**

e Suggest how the modelling assumption that the resistances to motion are constant could be refined to be more realistic. **(1)**

← Section 2.4

E **21** A car of mass 1000 kg is moving along a straight road with constant acceleration a m s^{-2}. The resistance to motion is modelled as a constant force of magnitude 1200 N. When the car is travelling at 12 m s^{-1}, the power generated by the engine of the car is 24 kW.

a Calculate the value of a. **(4)**

When the car is travelling at $14\,\text{m}\,\text{s}^{-1}$, the engine is switched off and the car comes to rest without braking in a distance d metres.

b Assuming the same model for resistance, use the work–energy principle to calculate the value of d. **(3)**

c Give a reason why the model used for resistance may not be realistic. **(1)**

← Sections 2.3, 2.4

E **22**

The figure shows the path taken by a cyclist in travelling on a section of a road. When the cyclist comes to the point A on the top of the hill she is travelling at $8\,\text{m}\,\text{s}^{-1}$. She descends a vertical distance of 20 m to the bottom of the hill. The road then rises to the point B through a vertical distance of 12 m. When she reaches B her speed is $5\,\text{m}\,\text{s}^{-1}$. The total mass of the cyclist and the bicycle is 80 kg and the total distance along the road from A to B is 500 m. By modelling the resistance to the motion of the cyclist as of constant magnitude 20 N,

a find the work done by the cyclist in moving from A to B. **(5)**

At B the road is horizontal.

b Given that at B the cyclist is accelerating at $0.5\,\text{m}\,\text{s}^{-2}$, find the power generated by the cyclist at B. **(4)**

← Sections 2.3, 2.4

E **23** A car of mass 400 kg is moving up a straight road inclined at an angle θ to the horizontal where $\sin\theta = \frac{1}{14}$. The resistance to motion of the car from non-gravitational forces is modelled as a constant force of magnitude R newtons.

When the car is moving at a constant speed of $20\,\text{m}\,\text{s}^{-1}$, the power developed by the car's engine is 10 kW.

Find the value of R. **(5)**

← Section 2.4

E **24** A lorry of mass 1500 kg moves along a straight horizontal road. The resistance to motion of the lorry has magnitude 750 N and the lorry's engine is working at a rate of 36 kW.

a Find the acceleration of the lorry when its speed is $20\,\text{m}\,\text{s}^{-1}$. **(4)**

The lorry comes to a hill inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{10}$ The magnitude of the resistance to motion from non-gravitational forces remains 750 N.

The lorry moves up the hill at a constant speed of $20\,\text{m}\,\text{s}^{-1}$.

b Find the rate at which the lorry is now working. **(3)**

← Section 2.4

E **25** A car of mass 1200 kg moves along a straight horizontal road. The resistance to motion of the car from non-gravitational forces is of constant magnitude 600 N. The car moves with constant speed and the engine of the car is working at a rate of 21 kW.

a Find the speed of the car. **(2)**

The car moves up a hill inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{14}$

The car's engine continues to work at 21 kW and the resistance to motion from non-gravitational forces remains of magnitude 600 N.

b Find the constant speed at which the car moves up the hill. **(4)**

← Section 2.4

E **26** A car of mass 1000 kg is moving along a straight horizontal road. The resistance to motion is modelled as a constant force of magnitude R newtons. The engine of the car is working at a constant rate of 12 kW. When the car is moving with speed 15 m s^{-1}, the acceleration of the car is 0.2 m s^{-2}.

a Show that $R = 600$. **(4)**

The car now moves with constant speed U m s^{-1} downhill on a straight road inclined at θ to the horizontal, where $\sin\theta = \frac{1}{40}$. The engine of the car is now working at a rate of 7 kW. The resistance to motion from non-gravitational forces remains of magnitude R newtons.

b Calculate the value of U. **(5)**

← Section 2.4

E/P **27** A motorcycle of mass 600 kg moves along a straight road at a speed of v m s^{-1}. The total resistances to motion of the motorcycle are modelled as a variable force of magnitude $(500 + 2v^2)$ N.

Calculate the power that must be generated by the motorcycle engine to maintain a constant speed of 15 m s^{-1}

a when the road is horizontal **(4)**

b when the road slopes downhill at an angle of 5° to the horizontal. **(5)**

← Section 2.4

E/P **28** A van of mass 1500 kg is driving up a straight road inclined at angle α to the horizontal, where $\sin\alpha = \frac{1}{12}$. The resistance to motion due to non-gravitational forces is modelled as a variable force of magnitude $(700 + 10v)$ N, where v m s^{-1} is the speed of the van.

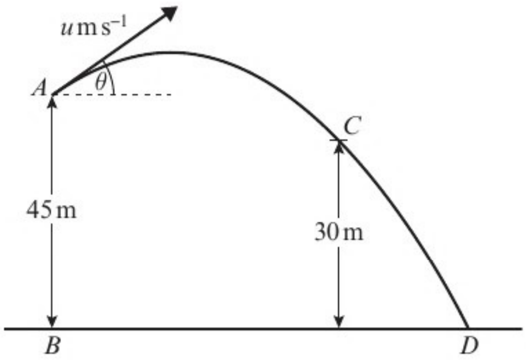
a Given that initially the speed of the van is 30 m s^{-1} and that the van's engine is working at a rate of 60 kW, calculate the magnitude of the initial deceleration of the van. **(4)**

When travelling up the same hill, the rate of working of the van's engine is increased to 80 kW.

b Using the same model for the resistance due to non-gravitational forces, calculate in m s^{-1} the maximum constant speed which can be sustained by the van at this rate of working. **(5)**

← Section 2.4

A E **29**



A particle P is projected from a point A with speed u m s^{-1} at an angle of elevation θ, where $\cos\theta = \frac{4}{5}$. The point B, on horizontal ground, is vertically below A and $AB = 45$ m. After projection, P moves freely under gravity, passing through a point C, 30 m above the horizontal ground, before striking the ground at the point D, as shown in the figure above.

Given that P passes through C with speed 24.5 m s^{-1},

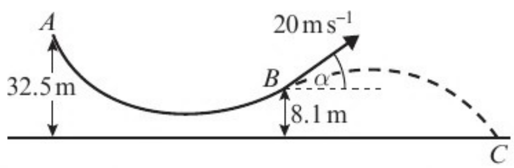
a using conservation of energy, or otherwise, show that $u = 17.5$ **(4)**

b find the size of the angle which the velocity of P makes with the horizontal as P passes through C **(7)**

c find the distance BD. **(3)**

← Section 2.3

E **30**



In a ski jumping competition, a skier of mass 80 kg moves from rest at a point A on a ski slope. The skier's path is an arc AB. The starting point A of the slope is 32.5 m above horizontal ground. The end B of the slope is 8.1 m above the ground. When the skier reaches B she is travelling

A

at $20\,\text{m}\,\text{s}^{-1}$ and moving upwards at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$, as shown in the figure. The distance along the slope from A to B is 60 m. The resistance to motion while she is on the slope is modelled as a force of constant magnitude R newtons.

a By using the work–energy principle, find the value of R. **(5)**

On reaching B, the skier then moves through the air and reaches the ground at the point C. The motion of the skier in moving from B to C is modelled as that of a particle moving freely under gravity.

b Find the time the skier takes to move from B to C. **(5)**

c Find the horizontal distance from B to C. **(2)**

d Find the speed of the skier immediately before she reaches C. **(4)**

← Section 2.3

E/P **31** A particle P lies in equilibrium on a horizontal smooth surface, and is attached to points A and B on the same surface by means of two springs. Spring AP has natural length 0.8 m and modulus of elasticity 24 N, and spring PB has natural length 0.4 m and modulus of elasticity 20 N. Given that APB is a straight line and that the distance AB is 1.6 m, find:

a the distance AP **(6)**

b the tension in each spring. **(2)**

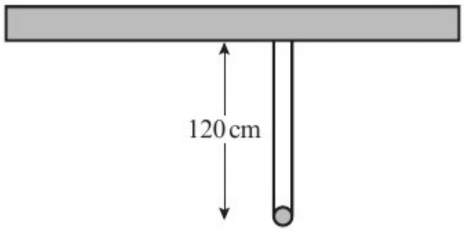
← Section 3.1

E/P **32** The elastic springs AB and BC are joined at B to form one long spring. The ends of the long spring are attached to two fixed points 4 m apart. The spring AB has natural length 1.5 m and modulus of elasticity 20 N, and the spring BC has natural length 0.75 m and modulus of elasticity 15 N.

Find the lengths AB and BC. **(8)**

← Section 3.1

A E **33**

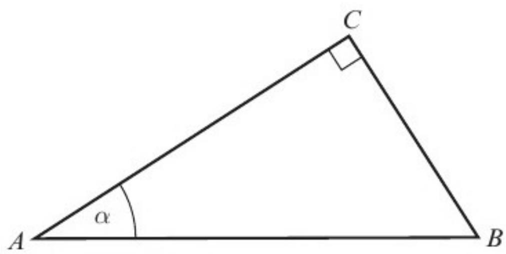


A particle of mass 5 kg is attached to one end of two light elastic strings. The other ends of the strings are attached to a hook on a beam. The particle hangs in equilibrium at a distance 120 cm below the hook with both strings vertical, as shown in the figure. One string has natural length 100 cm and modulus of elasticity 175 N. The other string has natural length 90 cm and modulus of elasticity λ newtons.

Find the value of λ. **(5)**

← Section 3.1

E **34**



A rod AB, of mass $2m$ and length $2a$, is suspended from a fixed point C by two light strings AC and BC. The rod rests horizontally in equilibrium with AC making an angle α with the rod, where $\tan\alpha = \frac{3}{4}$, and with AC perpendicular to BC, as shown in the figure.

a Give a reason why the rod cannot be uniform. **(1)**

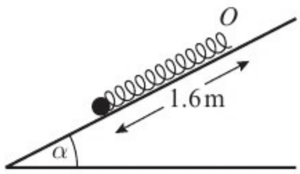
b Show that the tension in BC is $\frac{8}{5}mg$ and find the tension in AC. **(5)**

The string BC is elastic, with natural length a and modulus of elasticity kmg, where k is a constant.

c Find the value of k. **(4)**

← Section 3.1

A E **35**



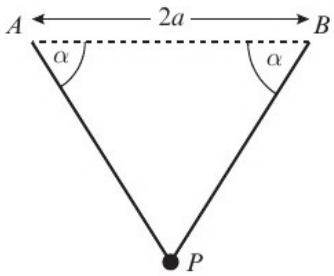
A particle of mass 0.8 kg is attached to one end of a light elastic spring, of natural length 2 m and modulus of elasticity 20 N. The other end of the spring is attached to a fixed point O on a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$.

The particle is held at a point which is 1.6 m down the line of greatest slope of the plane from O, as shown in the figure. The particle is then released from rest.

Find the initial acceleration of the particle. **(6)**

← Section 3.2

E/P **36**



Two light elastic strings each have natural length a and modulus of elasticity λ.
A particle P of mass m is attached to one end of each string. The other ends of the string are attached to points A and B, where AB is horizontal and $AB = 2a$. The particle is held at the midpoint of AB and released from rest. It comes to rest for the first time in its subsequent motion when PA and PB make angles α with AB, where $\tan\alpha = \frac{4}{3}$, as shown in the figure. Find λ in terms of m and g. **(7)**

← Section 3.2

E/P **37** One end of a light elastic string, of natural length 2 m and modulus of elasticity 19.6 N, is attached to a fixed point A. A small ball B of mass 0.5 kg is attached to the other end of the string.

A

The ball is released from rest at A and first comes to instantaneous rest at the point C, vertically below A.

a Find the distance AC. **(6)**

b Find the instantaneous acceleration of B at C. **(3)**

← Section 3.2

E/P **38** A light elastic string AB of natural length 1.5 m has modulus of elasticity 20 N. The end A is fixed to a point on a smooth horizontal table. A small ball S of mass 0.2 kg is attached to the end B. Initially S is at rest on the table with $AB = 1.5$ m. The ball S is then projected horizontally directly away from A with a speed of 5 m s^{-1}. By modelling S as a particle,

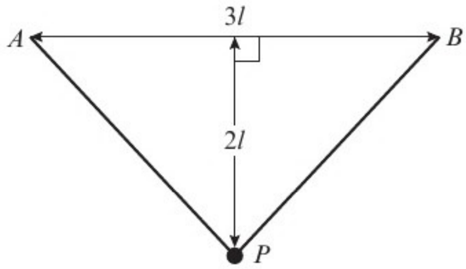
a find the speed of S when $AS = 2$ m. **(5)**

When the speed of S is 1.5 m s^{-1}, the string breaks.

b Find the tension in the string immediately before the string breaks. **(5)**

← Section 3.2

E/P **39**



A light elastic string, of natural length $3l$ and modulus of elasticity λ, has ends attached to two points A and B where $AB = 3l$ and AB is horizontal. A particle P of mass m is attached to the midpoint of the string. Given that P rests in equilibrium at a distance $2l$ below AB, as shown in the figure,

a show that $\lambda = \dfrac{15mg}{16}$ **(9)**

The particle is pulled vertically downwards from its equilibrium position until the total length of the elastic string is $7.8l$. The particle is released from rest.

A

b Show that P comes to instantaneous rest on the line AB. **(6)**

← Sections 3.2, 3.3

E/P **40** One end of an elastic string of natural length l m and modulus of elasticity λ N is fixed to a ceiling. A particle of mass m kg is attached to the free end and hangs in equilibrium. Show that the elastic potential energy stored by the string is given by $\frac{m^2g^2l}{2\lambda}$ **(5)**

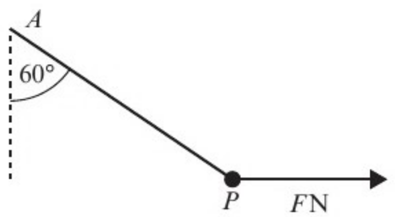
← Section 3.3

E/P **41** An elastic string has natural length 0.5 m and modulus of elasticity 20 N. One end of the string is fixed. A particle of mass 0.5 kg is attached to the free end and hangs in equilibrium. The string is then stretched to a length of 1 m.

Calculate the work done in stretching the string. **(6)**

← Section 3.3

E **42**



A particle of mass 0.8 kg is attached to one end of a light elastic string, of natural length 1.2 m and modulus of elasticity 24 N. The other end of the string is attached to a fixed point A. A horizontal force of magnitude F newtons is applied to P. The particle is in equilibrium with the string making an angle 60° with the downward vertical as shown in the figure. Calculate:

a the value of F **(3)**

b the extension of the string **(3)**

c the elastic energy stored in the string. **(2)**

← Sections 3.3, 3.4

A E/P **43** A particle P of mass m is attached to one end of a light elastic string, of natural length a and modulus of elasticity $3.6mg$. The other end of the string is fixed at a point O on a rough horizontal table. The particle is projected along the surface of the table from O with speed $\sqrt{2ag}$. At its furthest point from O, the particle is at the point A, where $OA = \frac{4}{3}a$.

a Find, in terms of m, g and a, the elastic energy stored in the string when P is at A. **(3)**

b Using the work–energy principle, or otherwise, find the coefficient of friction between P and the table. **(6)**

← Sections 3.3, 3.4

E/P **44** A particle P of mass m is held at a point A on a rough horizontal plane. The coefficient of friction between P and the plane is $\frac{2}{3}$. The particle is attached to one end of a light elastic string, of natural length a and modulus of elasticity $4mg$. The other end of the string is attached to a fixed point O on the plane, where $OA = \frac{3}{2}a$. The particle P is released from rest and comes to rest at a point B, where $OB < a$.

Using the work–energy principle, or otherwise, calculate the distance AB. **(6)**

← Sections 3.3, 3.4

E/P **45** A particle of mass 5 kg is attached to one end of a spring of natural length 1 m and modulus of elasticity 75 N. The other end of the spring is fixed to a point, P, on a smooth horizontal table. The particle is held 1.5 m from P and then released.

Show that the speed of the particle when the spring reaches its natural length is

$\frac{\sqrt{15}}{2}$ m s^{-1}. **(4)**

← Section 3.4

A E/P **46** A light elastic string, of natural length 0.8 m and modulus of elasticity 15 N has one end attached to a fixed point P. A particle of mass 0.5 kg is attached to the other end. The particle is held at a point which is 2 m vertically below P and released from rest. Find the speed of the particle:

a when the string first becomes slack **(4)**

b when the particle reaches P. **(2)**

← Section 3.4

E/P **47** A light elastic string has natural length 4 m and modulus of elasticity 58.8 N. A particle P of mass 0.5 kg is attached to one end of the string. The other end of the string is attached to a fixed point A. The particle is released from rest at A and falls vertically.

a Find the distance travelled by P before it comes to instantaneous rest for the first time. **(7)**

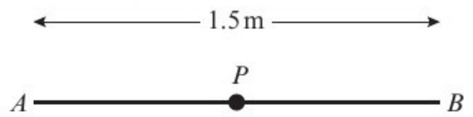
The particle is now held at a point 7 m vertically below A and released from rest.

b Find the speed of the particle when the string first becomes slack. **(5)**

← Section 3.4

E/P **48** Two light elastic strings each have natural length 0.75 m and modulus of elasticity 49 N. A particle P of mass 2 kg is attached to one end of each string. The other ends of the strings are attached to fixed points A and B, where AB is horizontal and $AB = 1.5$ m.

The particle is held at the midpoint of AB. The particle is released from rest, as shown below.



a Find the speed of P when it has fallen a distance of 1 m. **(6)**

A Given instead that P hangs in equilibrium vertically below the midpoint of AB with $\angle APB = 2\alpha$,

b show that $\tan\alpha + 5\sin\alpha = 5$. **(6)**

← Section 3.4

Challenge

1 The International Space Station (ISS) orbits the Earth at a height of 405 km above sea level, and has a mass of 420 000 kg.

A student estimates the work done in raising the ISS into orbit using the work-energy principle:

$$mgh = 420\,000 \times 9.8 \times 405\,000 = 1.7 \times 10^{12}\text{ J}$$

a Explain why the model used by this student is not suitable, and state whether the actual value work done will be greater or less than the value calculated by the student.

The force due to gravity, F, for an object of mass m kg a distance r m from the Earth's centre can be modelled using the formula

$F = (3.99 \times 10^{14})\dfrac{m}{r^2}$

b Given that radius of the earth is 6380 km, show that according to this model the work done in raising the ISS to its orbital height is 1.57×10^{12} J. ← Section 2.3

2 a Using integration, show that the work done in stretching a light elastic string of natural length l and modulus of elasticity λ, from length l to length $(l + x)$ is $\dfrac{\lambda x^2}{2l}$

b The same string is stretched from a length $(l + a)$ to a length $(l + b)$ where $b \geqslant a$. Show that the work done is the product of the mean tension and the distance moved.

← Sections 2.3, 3.2

A **3** One end of an elastic string of natural length 0.8 m and modulus of elasticity 120 N, is fixed to a point P. A particle of mass 1.2 kg is attached to the other end. The particle is initially held at rest at P and then released. Work out the distance the particle falls before it comes instantaneously to rest. ← Section 3.4

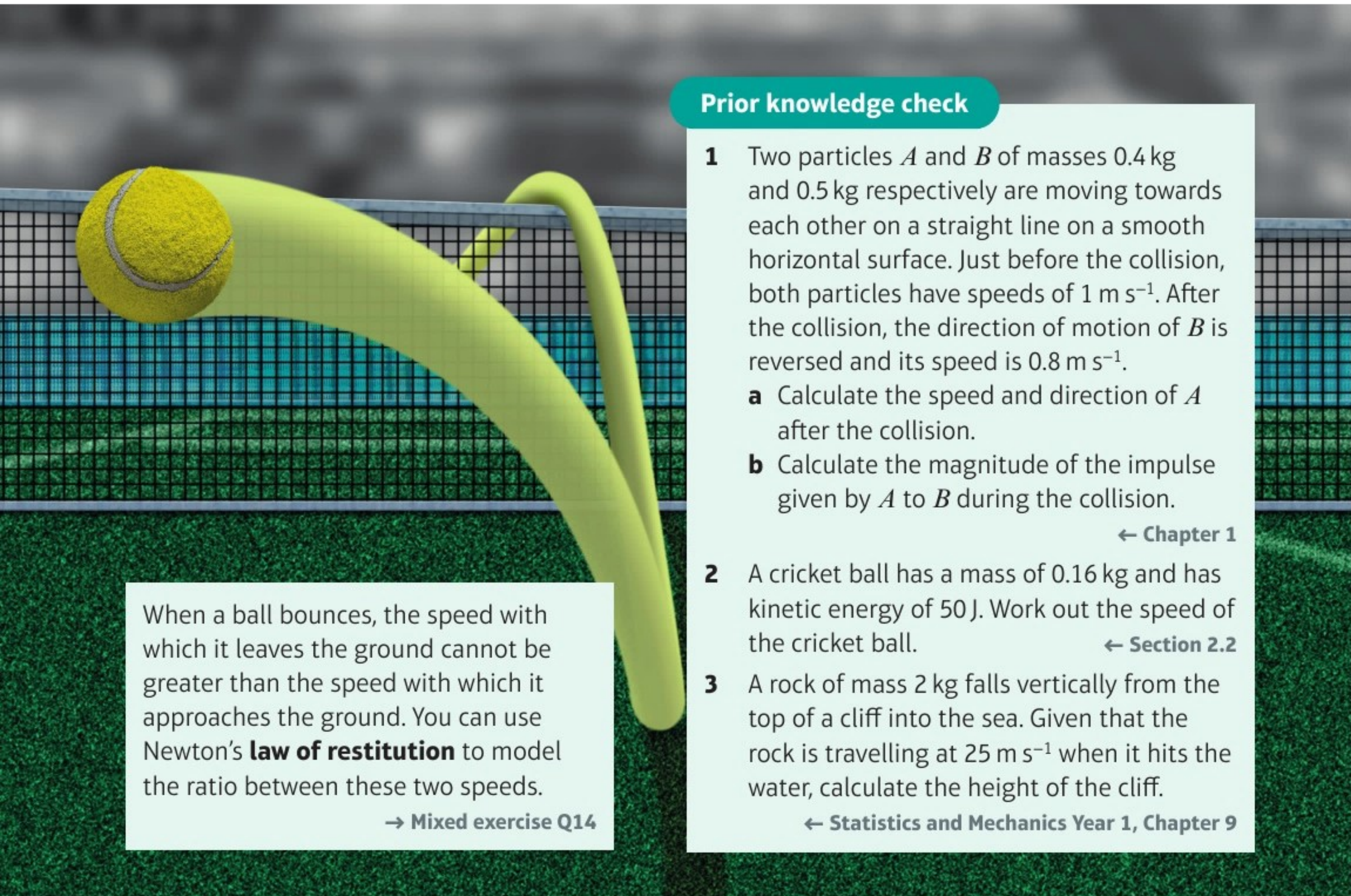
Elastic collisions in one dimension

4

Objectives

After completing this chapter you should be able to:

- Solve problems involving the direct impact of two particles by using the principle of conservation of momentum and Newton's law of restitution → pages 70–76
- Apply Newton's law of restitution to problems involving the direct collision of a particle with a smooth plane surface → pages 76–79
- Find the change in energy due to an impact or the application of an impulse → pages 79–84
- Solve problems involving successive direct impacts → pages 84–91



When a ball bounces, the speed with which it leaves the ground cannot be greater than the speed with which it approaches the ground. You can use Newton's **law of restitution** to model the ratio between these two speeds.

→ Mixed exercise Q14

Prior knowledge check

1 Two particles A and B of masses 0.4 kg and 0.5 kg respectively are moving towards each other on a straight line on a smooth horizontal surface. Just before the collision, both particles have speeds of 1 m s^{-1}. After the collision, the direction of motion of B is reversed and its speed is 0.8 m s^{-1}.

a Calculate the speed and direction of A after the collision.

b Calculate the magnitude of the impulse given by A to B during the collision.

← Chapter 1

2 A cricket ball has a mass of 0.16 kg and has kinetic energy of 50 J. Work out the speed of the cricket ball. ← Section 2.2

3 A rock of mass 2 kg falls vertically from the top of a cliff into the sea. Given that the rock is travelling at 25 m s^{-1} when it hits the water, calculate the height of the cliff.

← Statistics and Mechanics Year 1, Chapter 9

4.1 Direct impact and Newton's law of restitution

You can solve problems involving the direct impact of two particles by using the principle of conservation of momentum and **Newton's law of restitution**.

A direct impact is a collision between particles which are moving along the same straight line. When two particles collide, their speeds after the collision depend upon the materials from which they are made.

Newton's law of restitution (sometimes called Newton's experimental law) defines how the speeds of the particles after the collision depend on the nature of the particles as well as their speeds before the collision. This law only holds when the collision takes place in free space or on a smooth surface.

- **Newton's law of restitution states that**

$$\frac{\textbf{speed of separation of particles}}{\textbf{speed of approach of particles}} = e$$

The constant e is the coefficient of restitution between the particles. $0 \leqslant e \leqslant 1$

The value of the coefficient of restitution e depends on the materials from which the particles are made. In a perfectly elastic collision, $e = 1$ so the speed of separation is the same as the speed of approach. In a totally inelastic collision $e = 0$ so the particles coalesce on impact.

Notation **Coalesce** means join together. Two balls of sticky plasticine would produce a totally inelastic collision, with $e = 0$.

	Perfectly elastic collision (e = 1)	**Totally inelastic collision (e = 0)**
Before collision:	→ v m s^{-1} ←	→ v m s^{-1} ←
After collision:	← v m s^{-1} →	0 m s^{-1}

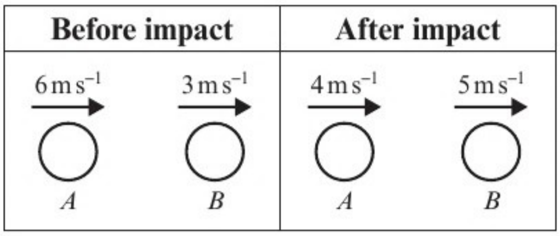
Example 1

In each part of this question, the two diagrams show the speeds and directions of motion of two particles A and B just before and just after a collision. The particles move on a smooth horizontal plane. Find the coefficient of restitution e in each case.

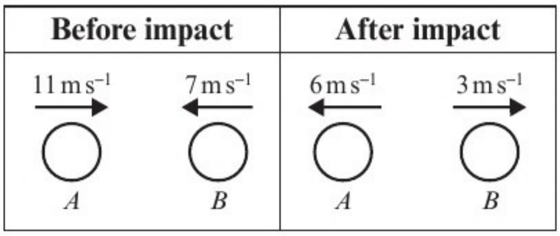
a

Before impact		After impact	
8 m s^{-1} →	At rest	At rest	2 m s^{-1} →
A	B	A	B

b



c



a The speed of approach is $8 - 0 = 8\text{ m s}^{-1}$

The speed of separation is $2 - 0 = 2\text{ m s}^{-1}$

$e = \frac{2}{8} = \frac{1}{4}$

b The speed of approach is $6 - 3 = 3\text{ m s}^{-1}$

The speed of separation is $5 - 4 = 1\text{ m s}^{-1}$

$e = \frac{1}{3}$

c The speed of approach is $11 + 7 = 18\text{ m s}^{-1}$

The speed of separation is $6 + 3 = 9\text{ m s}^{-1}$

$e = \frac{9}{18} = \frac{1}{2}$

Find the difference in the velocities before impact, called the speed of approach.

Find the difference in the velocities after impact, called the speed of separation.

Find e using

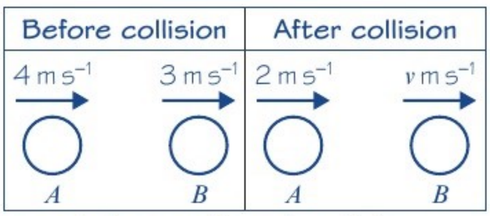
$$e = \frac{\text{speed of separation of particles}}{\text{speed of approach of particles}}$$

Watch out In part **c** the particles are moving in **opposite directions** so the speed of approach/separation will be the sum of the speeds of each particle.

Example 2

Two particles A and B are travelling in the same direction on a smooth surface with speeds 4 m s^{-1} and 3 m s^{-1} respectively. They collide directly, and immediately after the collision continue to travel in the same direction with speeds 2 m s^{-1} and $v\text{ m s}^{-1}$ respectively.

Given that the coefficient of restitution between A and B is $\frac{1}{3}$, find v.



$$\frac{\text{speed of separation of particles}}{\text{speed of approach of particles}} = e$$

$$\frac{v - 2}{4 - 3} = \frac{1}{3}$$

$$v - 2 = \frac{1}{3}$$

So $v = \frac{7}{3}$

Substitute the speed of approach, $4 - 3$, and the speed of separation, $v - 2$, then make v the subject of the formula.

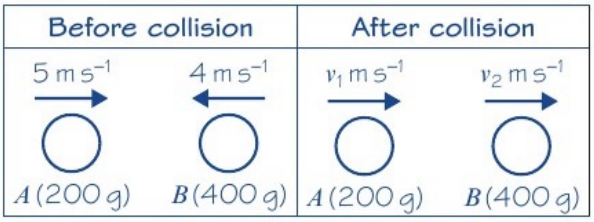
You can use the **principle of conservation of linear momentum** together with Newton's law of restitution to solve problems involving two unknown velocities.

Links For particles of masses m_1 and m_2 colliding in a straight line, with initial velocities u_1 and u_2 respectively, and final velocities v_1 and v_2 respectively:
$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ ← Section 1.2

Example 3

Two particles A and B of masses 200 g and 400 g respectively are travelling in opposite directions towards each other on a smooth surface with speeds 5 m s^{-1} and 4 m s^{-1} respectively. They collide directly, and immediately after the collision have velocities $v_1\text{ m s}^{-1}$ and $v_2\text{ m s}^{-1}$ respectively, measured in the direction of motion of A before the collision.

Given that the coefficient of restitution between A and B is $\frac{1}{2}$, find v_1 and v_2.



Online Explore direct impact using GeoGebra.

Draw a diagram showing the situation before and after the collision.

Using conservation of linear momentum for the system (→):

$$0.2 \times 5 + 0.4 \times (-4) = 0.2v_1 + 0.4v_2$$
$$1 - 1.6 = 0.2v_1 + 0.4v_2$$
$$-0.6 = 0.2v_1 + 0.4v_2$$
$$-3 = v_1 + 2v_2 \quad (1)$$

Problem-solving The final velocities are measured in the direction of motion of A before the collision, so choose this as the positive direction. Initially, B is moving in the opposite direction, so l_2 will be negative.

$$\frac{\text{speed of separation of particles}}{\text{speed of approach of particles}} = e$$
$$\frac{v_2 - v_1}{5 + 4} = \frac{1}{2}$$

Calculate the speed of approach and the speed of separation and substitute into Newton's law of restitution.

$$v_2 - v_1 = \frac{9}{2} \quad (2)$$

Eliminating v_1 between equations (1) and (2) gives

$v_2 = \frac{1}{2}$

Solve the simultaneous equations (1) and (2) to find the values of v_1 and v_2.

Substituting this value into equation (1) gives

$v_1 = -4$

Example 4

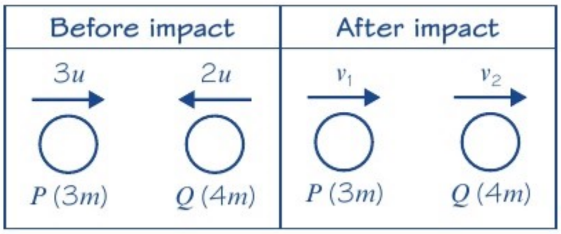
Two balls P and Q have masses $3m$ and $4m$ respectively. They are moving in opposite directions towards each other along the same straight line on a smooth level floor. Immediately before they collide, P has speed $3u$ and Q has speed $2u$. The coefficient of restitution between P and Q is e. By modelling the balls as smooth spheres and the floor as a smooth horizontal plane,

a show that the speed of Q after the collision is $\frac{u}{7}(15e + 1)$.

b Given that the direction of motion of P is unchanged, find the range of possible values of e.

c Given that the magnitude of the impulse of P on Q is $\frac{80mu}{9}$, find the value of e.

a



Using conservation of linear momentum for the system (→):

$9mu - 8mu = 3mv_1 + 4mv_2$

$\Rightarrow 3v_1 + 4v_2 = u \qquad (1)$

Newton's law of restitution gives

$\frac{v_2 - v_1}{3u + 2u} = e$

$\Rightarrow v_2 - v_1 = 5eu \qquad (2)$

Eliminating v_1 between equations (1) and (2) gives

$7v_2 = 15ue + u$

$v_2 = \frac{u}{7}(15e + 1)$

b Substituting this value into equation (2) gives

$v_1 = \frac{u}{7}(15e + 1) - 5eu$

$v_1 = \frac{u}{7}(1 - 20e)$

As $v_1 > 0$,

$\frac{u}{7}(1 - 20e) > 0$

So $e < \frac{1}{20}$

c Impulse of P on Q = change in momentum of Q

$= 4mv_2 - 4m(-2u)$

$= \frac{4mu}{7}(15e + 1) + 8mu$

$= \frac{60mu}{7}(1 + e)$

However the impulse is given as $\frac{80mu}{9}$

So $\frac{60mu}{7}(1 + e) = \frac{80mu}{9}$

$(1 + e) = \frac{28}{27}$

$e = \frac{1}{27}$

Online Explore direct impact with a known impulse using GeoGebra.

Choose a positive direction and draw a diagram showing masses and velocities before and after the impact. Use v_1 and v_2 for the unknown velocities after impact.

Use $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$, noting that u_2 is negative as Q is moving in the opposite direction.

Calculate the speed of approach and the speed of separation then substitute into Newton's law of restitution.

Solve the simultaneous equations (1) and (2) to find the value of v_2.

Now find the value of v_1 by substituting the value of v_2 from part **a** into equation (2).

Problem-solving

As the direction of motion of P is unchanged by the impact, v_1 must be positive.

This is $m_2v_2 - m_2u_2$

Exercise 4A

1 In each part of this question, the two diagrams show the speeds and directions of motion of two particles A and B just before and just after a collision. The particles move on a smooth horizontal plane. Find the coefficient of restitution e in each case.

	Before collision		After collision	
a	$6\,\text{m}\,\text{s}^{-1}$ → *A*	At rest *B*	At rest *A*	$4\,\text{m}\,\text{s}^{-1}$ → *B*
b	$4\,\text{m}\,\text{s}^{-1}$ → *A*	$2\,\text{m}\,\text{s}^{-1}$ → *B*	$2\,\text{m}\,\text{s}^{-1}$ → *A*	$3\,\text{m}\,\text{s}^{-1}$ → *B*
c	$9\,\text{m}\,\text{s}^{-1}$ → *A*	$6\,\text{m}\,\text{s}^{-1}$ ← *B*	$3\,\text{m}\,\text{s}^{-1}$ ← *A*	$2\,\text{m}\,\text{s}^{-1}$ → *B*

2 In each part of this question, the two diagrams show the speeds and directions of motion of two particles *A* and *B* just before a collision, and their velocities relative to the initial directon of *A* just after a collision. The particles move on a smooth horizontal plane. The masses of *A* and *B* and the coefficients of restitution *e* are also given. Find the values of v_1 and v_2 in each case.

	Before collision		After collision	
a $e = \frac{1}{2}$	$6\,\text{m}\,\text{s}^{-1}$ → *A* (0.25 kg)	At rest *B* (0.5 kg)	$v_1\,\text{m}\,\text{s}^{-1}$ → *A* (0.25 kg)	$v_2\,\text{m}\,\text{s}^{-1}$ → *B* (0.5 kg)
b $e = 0.25$	$4\,\text{m}\,\text{s}^{-1}$ → *A* (2 kg)	$2\,\text{m}\,\text{s}^{-1}$ → *B* (3 kg)	$v_1\,\text{m}\,\text{s}^{-1}$ → *A* (2 kg)	$v_2\,\text{m}\,\text{s}^{-1}$ → *B* (3 kg)
c $e = \frac{1}{7}$	$8\,\text{m}\,\text{s}^{-1}$ → *A* (3 kg)	$6\,\text{m}\,\text{s}^{-1}$ ← *B* (1 kg)	$v_1\,\text{m}\,\text{s}^{-1}$ → *A* (3 kg)	$v_2\,\text{m}\,\text{s}^{-1}$ → *B* (1 kg)
d $e = \frac{2}{3}$	$6\,\text{m}\,\text{s}^{-1}$ → *A* (400 g)	$6\,\text{m}\,\text{s}^{-1}$ ← *B* (400 g)	$v_1\,\text{m}\,\text{s}^{-1}$ → *A* (400 g)	$v_2\,\text{m}\,\text{s}^{-1}$ → *B* (400 g)
e $e = \frac{1}{5}$	$3\,\text{m}\,\text{s}^{-1}$ → *A* (5 kg)	$12\,\text{m}\,\text{s}^{-1}$ ← *B* (4 kg)	$v_1\,\text{m}\,\text{s}^{-1}$ → *A* (5 kg)	$v_2\,\text{m}\,\text{s}^{-1}$ → *B* (4 kg)

3 A small smooth sphere A of mass 1 kg is travelling along a straight line on a smooth horizontal plane with speed 4 m s^{-1} when it collides with a second smooth sphere B of the same radius, with mass 2 kg and travelling in the same direction as A with speed 2.5 m s^{-1}. After the collision, A continues in the same direction with speed 2 m s^{-1}. Find:

a the speed of B after the collision

b the coefficient of restitution for the spheres.

4 Two spheres A and B have masses 2 kg and 6 kg respectively. A and B move towards each other in opposite directions along the same straight line on a smooth horizontal surface with speeds 4 m s^{-1} and 6 m s^{-1} respectively. If the coefficient of restitution is $\frac{1}{5}$, find the velocities of the spheres after the collision and the magnitude of the impulse given to each sphere.

(P) **5** Two particles P and Q of masses $2m$ and $3m$ respectively are moving in opposite directions towards each other. Each particle is travelling with speed u. Given that Q is brought to rest by the collision, find the speed of P after the collision, and the coefficient of restitution between the particles.

(P) **6** Two particles A and B are travelling along the same straight line in the same direction on a smooth horizontal surface with speeds $3u$ and u respectively. Particle A catches up and collides with particle B. If the mass of B is twice that of A and the coefficient of restitution is e, find, in terms of e and u, expressions for the speeds of A and B after the collision.

(P) **7** Two identical particles of mass m are projected towards each other along the same straight line on a smooth horizontal surface with speeds of $2u$ and $3u$. After the collision, the directions of motion of both particles are reversed. Show that this implies that the coefficient of restitution e satisfies the inequality $e > \frac{1}{5}$

(E/P) **8** Two particles A and B of masses m and km respectively are placed on a smooth horizontal plane. Particle A is made to move on the plane with speed u so as to collide directly with B, which is at rest. After the collision B moves with speed $\frac{3}{10}u$.

a Find, in terms of u and the constant k, the speed of A after the collision. **(4 marks)**

b By using Newton's law of restitution, show that $\frac{7}{3} \leqslant k \leqslant \frac{17}{3}$ **(5 marks)**

> **Problem-solving**
> In part **b** use the limits of e to set up an inequality.

(E/P) **9** Two particles A and B of masses m and $3m$ respectively are placed on a smooth horizontal plane. Particle A is made to move on the plane with speed $2u$ so as to collide directly with B, which is moving in the same direction with speed u. After the collision B moves with speed ku, where k is a constant.

a Find, in terms of u and the constant k, the speed of A after the collision. **(4 marks)**

b By using Newton's law of restitution, show that $\frac{5}{4} \leqslant k \leqslant \frac{3}{2}$ **(5 marks)**

(E/P) **10** A particle P of mass m is moving with speed $4u$ on a smooth horizontal plane. The particle collides directly with a particle Q of mass $3m$ moving with speed $2u$ in the same direction as P. The coefficient of restitution between P and Q is e.

a Show that the speed of Q after the collision is $\frac{u}{2}(5 + e)$. **(6 marks)**

b Find the speed of P after the collision, giving your answer in terms of e. **(4 marks)**

c Show that the direction of motion of P is unchanged by the collision. **(2 marks)**

d Given that the magnitude of the impulse of P on Q is $2mu$, find the value of e. **(4 marks)**

Challenge

Two particles P and Q of masses $3m$ kg and m kg respectively move towards each other in a straight line in opposite directions on a smooth horizontal surface. P has initial speed 2 m s^{-1} and Q has initial speed u m s^{-1}. After P and Q collide, both particles move in the same direction as P's original motion. Given that, after the impact, Q moves with twice the speed of P and that the coefficient of restitution between P and Q is $\frac{1}{4}$, show that $u = \frac{14}{9}$

4.2 Direct collision with a smooth plane

You can also apply Newton's law of restitution to problems involving the direct collision of a particle with a smooth plane surface perpendicular to the direction of motion of the particle.

In the figure a particle is shown moving horizontally with speed u before impact with a vertical plane surface. After impact the particle moves in the opposite direction with speed v.

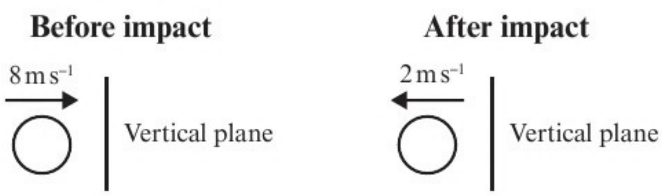
The speed of the particle after the impact depends on the speed of the particle before the impact and the coefficient of restitution e between the particle and the plane.

- **For the direct collision of a particle with a smooth plane, Newton's law of restitution can be written as**

$$\frac{\textbf{speed of rebound}}{\textbf{speed of approach}} = e$$

Example 5

A particle collides normally with a fixed vertical plane. The diagram shows the speeds of the particle before and after the collision. Find the value of the coefficient of restitution e.



Notation If a particle collides **normally** then its direction of motion immediately before the instant of impact is perpendicular to the plane.

Using Newton's law of restitution,

$$e = \frac{\text{speed of rebound}}{\text{speed of approach}}$$

$$= \frac{2}{8} = \frac{1}{4}$$

This is the same as $e = \frac{2-0}{8-0}$

The coefficient of restitution is $\frac{1}{4}$

Example 6

A small sphere collides normally with a fixed vertical wall. Before the impact the sphere is moving with a speed of $4\,\text{m s}^{-1}$ on a smooth horizontal floor. The coefficient of restitution between the sphere and the wall is 0.2. Find the speed of the sphere after the collision.

Using Newton's law of restitution,

$$e = \frac{\text{speed of rebound}}{\text{speed of approach}}$$

$$0.2 = \frac{v}{4}$$

Let the speed of the sphere after the collision be $v\,\text{m s}^{-1}$.

So $v = 0.8$

The speed of the sphere after the collision is $0.8\,\text{m s}^{-1}$.

Example 7

A particle falls 22.5 cm from rest onto a smooth horizontal plane. It then rebounds to a height of 10 cm. Find the coefficient of restitution between the particle and the plane. Give your answer to 2 significant figures.

Online Explore the direct collision of a falling particle with a smooth plane using GeoGebra.

As particle falls:

Use $v^2 = u^2 + 2as$

The particle is falling under gravity so use the appropriate constant acceleration formula to find its speed when it hits the plane.
← **Statistics and Mechanics Year 1, Section 9.5**

with $u = 0$, $s = 0.225$ and $a = g$

$v^2 = 0.45g$

$v = 2.1$

Using $g = 9.8\,\text{m s}^{-2}$, calculate $v\,\text{m s}^{-1}$, the speed of the particle when it hits the plane.

After impact:

Use $v^2 = u^2 + 2as$

with $v = 0$, $s = 0.1$ and $a = -g$

After it rebounds it initially moves upwards under gravity. As the upward direction is taken as positive here, the acceleration is negative.

$u^2 = 0.2g$

$u = 1.4$

$u\,\text{m s}^{-1}$ is the rebound speed of the particle.

Using Newton's law of restitution,

$$e = \frac{\text{speed of rebound}}{\text{speed of approach}}$$

$$= \frac{1.4}{2.1} = \frac{2}{3}$$

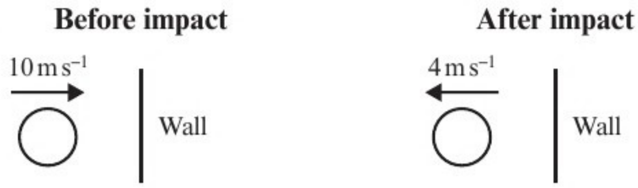
The coefficient of restitution is 0.67 (2 s.f.)

Exercise 4B

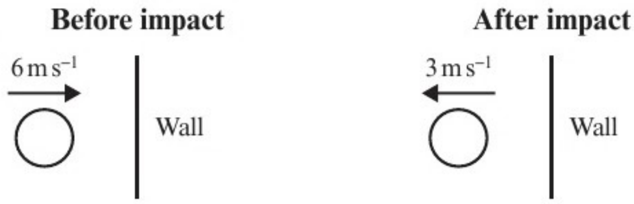
Whenever a numerical value of g is required, take $g = 9.8\,\mathrm{m\,s^{-2}}$.

1 A smooth sphere collides normally with a fixed vertical wall. The two diagrams show the speed and direction of motion of the sphere before and after the collision. In each case, find the value of the coefficient of restitution e.

a



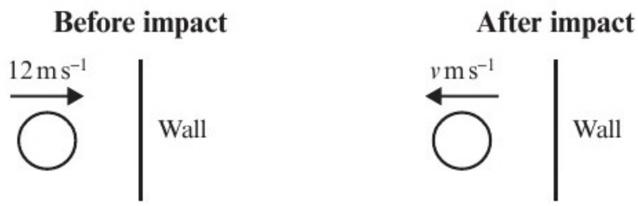
b



2 A smooth sphere collides normally with a fixed vertical wall. The two diagrams show the speed and direction of motion of the sphere before and after the collision. The value of e is given in each case. Find the speed of the sphere after the collision in each case.

a $e = \frac{1}{2}$

b $e = \frac{1}{4}$



3 A smooth sphere collides normally with a fixed vertical wall. The two diagrams show the speed and direction of motion of the sphere before and after the collision. The value of e is also given. Find the speed of the sphere before the collision in each case.

a $e = \frac{1}{2}$

b $e = \frac{3}{4}$

4 A small smooth sphere of mass 0.3 kg is moving on a smooth horizontal table with a speed of $10\,\text{m s}^{-1}$ when it collides normally with a fixed smooth wall. It rebounds with a speed of $7.5\,\text{m s}^{-1}$. Find the coefficient of restitution between the sphere and the wall.

5 A particle falls 2.5 m from rest onto a smooth horizontal plane. It then rebounds to a height of 1.5 m. Find the coefficient of restitution between the particle and the plane. Give your answer to 2 significant figures.

(E) 6 A particle falls 3 m from rest onto a smooth horizontal plane. It then rebounds to a height h m. The coefficient of restitution between the particle and the plane is 0.25.

a Find the value of h. **(4 marks)**

b Without further calculation, state how your answer to part **a** would change if $e > 0.25$ **(1 mark)**

(E/P) 7 A small smooth sphere falls from rest onto a smooth horizontal plane and rebounds from the plane. It takes 2 seconds to reach the plane then another 2 seconds to reach the plane a second time. Find the coefficient of restitution between the sphere and the plane. **(7 marks)**

(E/P) 8 A small smooth sphere falls from rest onto a smooth horizontal plane. It takes 3 seconds to reach the plane. The coefficient of restitution between the sphere and the plane is 0.49. Find the time it takes after rebound for the sphere to reach the plane a second time. **(7 marks)**

(E/P) 9 A particle falls from rest from a height of h m above level ground. After rebounding the particle reaches a maximum height of $\frac{1}{2}h$. Find, in terms of g and h, the height of the particle 1 second after it hits the ground. **(8 marks)**

Challenge

A particle P falls from rest from a height of h m above level ground. Show that after hitting the ground, the maximum height that the particle reaches is he^2, where e is the coefficient of restitution between the particle and the ground.

4.3 Loss of kinetic energy

You can solve problems that ask you to find the change in kinetic energy due to an impact or the application of an impulse.

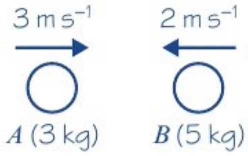
Example 8

Two spheres A and B have masses 3 kg and 5 kg respectively. A and B move towards each other in opposite directions along the same straight line on a smooth horizontal surface with speeds $3\,\text{m s}^{-1}$ and $2\,\text{m s}^{-1}$ respectively.

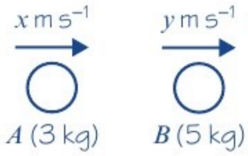
a Given the coefficient of restitution is $\frac{3}{5}$, find the velocities of the spheres after the collision.

b Find the loss of kinetic energy due to the impact.

a Before impact between A and B



After impact between A and B



Using conservation of momentum (→),

$3 \times 3 + 5 \times (-2) = 3 \times x + 5 \times y$

$\Rightarrow 3x + 5y = -1$ (1)

Using Newton's law of restitution,

$\frac{y - x}{3 + 2} = \frac{3}{5}$

$\Rightarrow y - x = 3$ (2)

Solving equations (1) and (2) gives

$y = 1$ and $x = -2$

After the impact the direction of A is reversed and its speed is 2 m s^{-1}. The direction of B is also reversed and its speed is 1 m s^{-1}.

b The total kinetic energy before impact is

$\frac{1}{2} \times 3 \times 3^2 + \frac{1}{2} \times 5 \times 2^2 = 23.5\text{ J}$

The total kinetic energy after impact is

$\frac{1}{2} \times 3 \times 2^2 + \frac{1}{2} \times 5 \times 1^2 = 8.5\text{ J}$

So the loss of kinetic energy is

$23.5\text{ J} - 8.5\text{ J} = 15\text{ J}$

Online Explore the loss of kinetic energy in a collision using GeoGebra.

Draw diagrams to show the masses, speeds and directions of A and B before and after the collision.

Let the velocity of A be $x\text{ m s}^{-1}$ and let the velocity of B be $y\text{ m s}^{-1}$ after the impact.

Use
$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

Use
$\frac{\text{speed of separation of particles}}{\text{speed of approach of particles}} = e$

Solve the simultaneous equations (1) and (2) to find x and y.

The total kinetic energy before impact is
$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$

The total kinetic energy after impact is
$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

- **The loss of kinetic energy due to impact is**
  $$\left(\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2\right) - \left(\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2\right)$$

Watch out When the particles are perfectly elastic ($e = 1$) you will find that there is no loss of kinetic energy due to impact.
In all practical situations $e < 1$ and some kinetic energy is converted into heat or sound energy at impact.

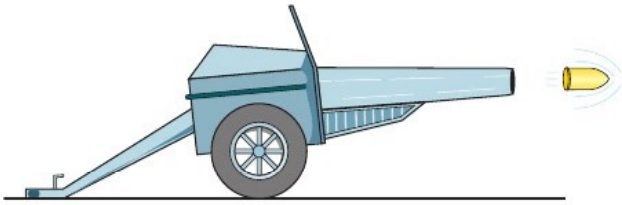
Example 9

A gun of mass 600 kg fires a shell of mass 12 kg horizontally with speed 200 m s^{-1}.

a Find the velocity of the gun after the shell has been fired.

b Find the total kinetic energy generated on firing.

c Show that the ratio of the energy of the gun to the energy of the shell is equal to the ratio of the speed of the gun to the speed of the shell after firing.



a Before firing 0 m s^{-1} → 0 m s^{-1} →

Draw diagrams to show the masses and velocities of the gun and the shell before and after firing.

After firing $x\text{ m s}^{-1}$ → 200 m s^{-1} →

Let the velocity of the gun be x m s^{-1}.

Using conservation of momentum (→),

$600 \times 0 + 12 \times 0 = 600 \times x + 12 \times 200$

$\Rightarrow x = -4$

When a gun is fired, the shell and the gun acquire equal and opposite momentum. So use $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

After the impact the direction of the gun is reversed and its speed is 4 m s^{-1}.

b The total kinetic energy after firing is

$\frac{1}{2} \times 600 \times 4^2 + \frac{1}{2} \times 12 \times 200^2$

$= 4800 + 240\,000$

$= 244\,800\text{ J}$

Both the gun and the shell acquire kinetic energy supplied by expanding gas, resulting from the chemical reaction that takes place on firing.

So the total kinetic energy generated on firing is 244 800 J.

c Ratio of K.E. of gun to K.E. of shell is

$\frac{1}{2} \times 600 \times 4^2 : \frac{1}{2} \times 12 \times 200^2$

$= 4800 : 240\,000$

$= 1 : 50$

The energy of the shell is much greater than the energy of the gun.

Ratio of speed of gun to speed of shell is $4 : 200 = 1 : 50$

Example 10

Two particles A and B, of masses 200 g and 300 g respectively, are connected by a light inextensible string. The particles are side by side at rest on a smooth floor and A is projected with speed 6 m s^{-1} directly away from B. When the string becomes taut, particle B is jerked into motion and A and B then move with a common speed in the direction of projection of A. Find:

a the common speed of the particles after the string becomes taut

b the loss of total kinetic energy due to the jerk.

a Before the jerk

After the jerk

Using conservation of momentum (→),

$0.3 \times 0 + 0.2 \times 6 = 0.3V + 0.2V$

$\Rightarrow V = 2.4$

So the common speed is $2.4\,\text{m s}^{-1}$.

b The total kinetic energy before the jerk is

$\frac{1}{2} \times 0.3 \times 0^2 + \frac{1}{2} \times 0.2 \times 6^2 = 3.6\,\text{J}$

The total kinetic energy after the jerk is

$\frac{1}{2} \times 0.3 \times 2.4^2 + \frac{1}{2} \times 0.2 \times 2.4^2 = 1.44\,\text{J}$

So the loss of kinetic energy is

$3.6 - 1.44 = 2.16\,\text{J}$

You will need to convert the units of mass into kg so that the units of energy are joules.

Problem-solving

When the string becomes taut, both particles experience equal and opposite impulses due to the tension in the string. Linear momentum is conserved during the jerk, so you can use $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

The total kinetic energy before the string becomes taut is $\frac{1}{2}m_1{u_1}^2 + \frac{1}{2}m_2{u_2}^2$.

The total kinetic energy after the string becomes taut is $\frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2$.

Exercise 4C

1 A particle A of mass 500 g lies at rest on a smooth horizontal table. A second particle B of mass 600 g is projected along the table with speed $6\,\text{m s}^{-1}$ and collides directly with A. If the collision reduces the speed of B to $1\,\text{m s}^{-1}$, without changing its direction, find:

a the speed of A after the collision

b the loss of kinetic energy due to the collision.

2 Two particles A and B of masses m and $2m$ respectively move towards each other in opposite directions with speeds u and $2u$. If the coefficient of restitution between the particles is $\frac{2}{3}$, find the velocities of A and of B after the collision. Find also, in terms of m and u, the loss of kinetic energy due to the collision.

3 A particle of mass 3 kg moving with speed $6\,\text{m s}^{-1}$ collides directly with a particle of mass 5 kg moving in the opposite direction with speed $2\,\text{m s}^{-1}$. The particles coalesce and move with velocity v after the collision. Find the loss of kinetic energy due to the impact.

4 A billiard ball of mass 200 g strikes a smooth cushion at right angles. Its speed before the impact is $2.5\,\text{m s}^{-1}$ and the coefficient of restitution is $\frac{4}{5}$. Find the loss in kinetic energy of the billiard ball due to the impact.

5 A bullet of mass 0.15 kg moving horizontally at $402\,\text{m s}^{-1}$ embeds itself in a sandbag of mass 30 kg, which is suspended freely. Assuming that the sandbag is stationary before the impact, find:

a the common velocity of the bullet and the sandbag after the impact

b the loss of kinetic energy due to the impact.

6 A bullet is fired horizontally from a rifle. The rifle has mass 4.8 kg and the bullet has mass 20 g. The initial speed of the bullet is 400 m s^{-1}. Find:

a the initial speed with which the rifle recoils

b the total kinetic energy generated as a result of firing the bullet.

(P) **7** A train of mass 30 tonnes moving with a small speed V impacts upon a number of stationary carriages each weighing 6 tonnes. The complete train and carriages now move forward with a speed of $\frac{5}{8}V$. Find:

a the number of stationary carriages

b the fraction of the original kinetic energy lost in the impact.

(E) **8** A truck of mass 5 tonnes is moving in a straight line at 1.5 m s^{-1} towards a second stationary truck of mass 10 tonnes which is at rest. The trucks collide, and after the impact the second truck moves at 0.6 m s^{-1}. Modelling the trucks as particles, find:

Hint 1 tonne = 1000 kg

a the velocity of the first truck after the impact **(3 marks)**

b the coefficient of restitution between the two trucks **(2 marks)**

c the loss of kinetic energy due to the impact. **(3 marks)**

(P) **9** A particle of mass m moves in a straight line with speed v when it explodes into two parts, one of mass $\frac{1}{3}m$ and the other of mass $\frac{2}{3}m$, both moving in the same direction as before. If the explosion increases the energy of the system by $\frac{1}{4}mu^2$, where u is a positive constant, find the speeds of the particles immediately after the explosion. Give your answers in terms of u and v.

(E/P) **10** A small smooth sphere A of mass 2 kg moves at 4 m s^{-1} on a smooth horizontal table. It collides directly with a second small smooth sphere B of mass 3 kg, which is moving in the same direction at a speed of 1 m s^{-1}. The loss of kinetic energy due to the collision is 3 J. After the collision A and B continue to move in the same direction with speeds u and v respectively.

a Show that $5v^2 - 22v + 21 = 0$. **(6 marks)**

b Hence find u and v, carefully justifying your choice of solutions. **(5 marks)**

(E) **11** Two particles A and B, of masses 2 kg and 5 kg respectively, are connected by a light inextensible string. The particles are side by side on a smooth floor and A is projected with speed 7 m s^{-1} directly away from B. When the string becomes taut, particle B is jerked into motion and A and B then move with a common speed in the direction of the original velocity of A. Find:

a the common speed of the particles after the string becomes taut **(4 marks)**

b the loss of total kinetic energy due to the jerk. **(3 marks)**

E/P **12** Two particles A and B, of masses m and M respectively, are connected by a light inextensible string. The particles are side by side on a smooth floor and A is projected with speed u directly away from B. When the string becomes taut, particle B is jerked into motion and A and B then move with a common speed in the direction of the original projection of A.

Show that the loss of total kinetic energy due to the jerk is $\frac{mMu^2}{2(m + M)}$ **(8 marks)**

Problem-solving

Start by finding an expression for the common speed, v, after the string becomes taut.

E **13** Two particles of masses 3 kg and 5 kg lie on a smooth table and are connected by a slack inextensible string. The first particle is projected along the table with a velocity of 20 m s^{-1} directly away from the second particle. Find:

a the velocity of each particle after the string has become taut **(6 marks)**

b the difference between the kinetic energies of the system when the string is slack and when it is taut. **(3 marks)**

E/P **14** Three small spheres of masses 20 g, 40 g and 60 g respectively lie in order in a straight line on a large smooth table. The distance between adjacent spheres is 10 cm. Two slack strings, each 70 cm in length, connect the first sphere with the second, and the second sphere with the third. The 60 g sphere is projected with a speed of 5 m s^{-1}, directly away from the other two. Find:

a the time which elapses before the 20 g sphere begins to move and the speed with which it starts **(8 marks)**

b the loss in kinetic energy resulting from the two jerks. **(5 marks)**

Challenge

Two small spheres A and B with masses 4 kg and 1 kg respectively lie on a large smooth surface. A and B are connected by a light inextensible string and are projected directly towards each other with speeds 2 m s^{-1} and 3 m s^{-1} respectively. The coefficient of restitution for the collision of A and B is 0.8. After the collision both particles move in the direction in which A was originally moving. Find the kinetic energy of the system when the string becomes taut.

4.4 Successive direct impacts

You can solve problems involving successive direct impacts of particles with each other, or with a smooth plane surface. When you are solving such problems, you should draw a clear diagram showing the 'before' and 'after' information for each collision.

Example 11

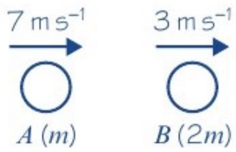
Three spheres A, B and C have masses m, $2m$ and $3m$ respectively. The spheres move along the same straight line on a horizontal plane with A following B, which is following C. Initially the speeds of A, B and C are $7\,\text{m s}^{-1}$, $3\,\text{m s}^{-1}$ and $1\,\text{m s}^{-1}$ respectively, in the direction ABC. Sphere A collides with sphere B and then sphere B collides with sphere C. The coefficient of restitution between A and B is $\frac{1}{2}$ and the coefficient of restitution between B and C is $\frac{1}{4}$

a Find the velocities of the three spheres after the second collision.

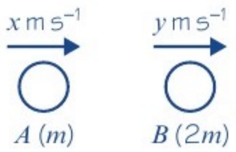
b Explain how you can predict that there will be a further collision between A and B.

a First collision:

Before impact between A and B



After impact between A and B



Using conservation of momentum (→),

$m \times 7 + 2m \times 3 = m \times x + 2m \times y$

$\Rightarrow\; x + 2y = 13 \qquad (1)$

Using Newton's law of restitution,

$\dfrac{y - x}{7 - 3} = \dfrac{1}{2}$

$\Rightarrow\; y - x = 2 \qquad (2)$

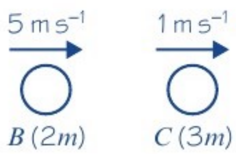
Solving equations (1) and (2) gives

$y = 5$ and $x = 3$

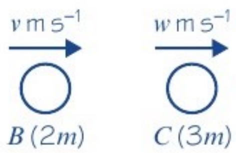
After the first impact, the velocity of A is $3\,\text{m s}^{-1}$ and the velocity of B is $5\,\text{m s}^{-1}$.

Second collision:

Before impact between B and C



After impact between B and C



Online Explore successive collisions using GeoGebra.

Draw diagrams to show the masses, speeds and directions of A and B before and after the first collision.

Let the velocity of A be $x\,\text{m s}^{-1}$ and let the velocity of B be $y\,\text{m s}^{-1}$ after the impact. Make it clear in your diagram which velocity corresponds to which particle and which direction is positive.

Use $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

Use $\dfrac{\text{speed of separation of particles}}{\text{speed of approach of particles}} = e$

Add equations (1) and (2) to eliminate x and to give $3y = 15$, then substitute $y = 5$ into equation (2) to give $x = 3$.

Draw diagrams to show the masses, speeds and directions of B and C before and after the collision.

Let the velocities of B and C after the collision be $v\,\text{m s}^{-1}$ and $w\,\text{m s}^{-1}$ respectively.

Using conservation of momentum (→),

$2m \times 5 + 3m \times 1 = 2m \times v + 3m \times w$

$\Rightarrow 2v + 3w = 13$ (3)

Using Newton's law of restitution,

$\frac{w - v}{5 - 1} = \frac{1}{4}$

$\Rightarrow w - v = 1$ (4)

Solving equations (3) and (4) gives

$w = 3$ and $v = 2$

Form two equations and solve them as you did for the first collision.

After the second impact, the velocity of B is $2\,\text{m s}^{-1}$ and the velocity of C is $3\,\text{m s}^{-1}$. The velocity of A is $3\,\text{m s}^{-1}$.

A was not involved in the second collision so its final velocity is $3\,\text{m s}^{-1}$.

b As the velocity of A is greater than, and in the same direction as, the velocity of B, there will be a further collision between A and B.

Problem-solving

A plane is assumed to extend infinitely in either direction. In real life, a particle will not continue to move in a straight line and at a constant speed forever.

Example 12

A uniform smooth sphere P of mass $3m$ is moving in a straight line with speed u on a smooth horizontal table. Another uniform smooth sphere Q of mass m is moving with speed $2u$ in the same straight line as P, but in the opposite direction. The sphere P collides with the sphere Q directly. The velocities of P and Q after the collision are v and w respectively, measured in the direction of motion of P before the collision. The coefficient of restitution between P and Q is e.

a Find expressions for v and w in terms of u and e.

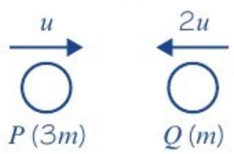
b Show that, if the direction of motion of P is changed by the collision, then $e > \frac{1}{3}$

Following the collision with P, the sphere Q then collides with and rebounds from a vertical wall, which is perpendicular to the direction of motion of Q. The coefficient of restitution between Q and the wall is e'.

c Given that $e = \frac{5}{9}$ and that P and Q collide again in the subsequent motion, show that $e' > \frac{1}{9}$

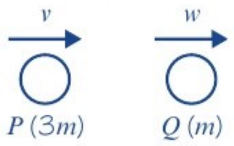
a First collision:

Before impact between P and Q



In your diagram ensure that the velocity of Q is in the opposite direction from that of P.

After impact between P and Q



Using conservation of momentum (→),

$3m \times u - m \times 2u = 3m \times v + m \times w$

> Each velocity must be given the correct sign in relation to the positive direction.

$\Rightarrow 3v + w = u$ (1)

Using Newton's law of restitution,

$\frac{w - v}{u + 2u} = e$

> Note that the spheres are initially moving in opposite directions, so the speed of approach is $u + 2u$.

$\Rightarrow w - v = 3eu$ (2)

Subtract equation (2) from equation (1) to give

> Solve the two simultaneous equations to obtain an expression for v.

$4v = u(1 - 3e)$

$v = \frac{u(1 - 3e)}{4}$

So $w = \frac{u(9e + 1)}{4}$

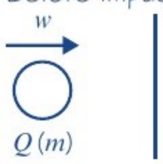
> Substitute v into equation (2) to find the expression for w.

b As $v < 0$, $\frac{u(1 - 3e)}{4} < 0$

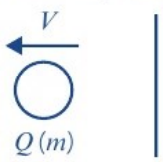
> If P changes direction, then v will be negative.

So $e > \frac{1}{3}$

c Before impact between Q and the wall



After impact between Q and the wall



> Draw diagrams for the new collision.

Using Newton's law of restitution,

$\frac{V}{w} = e'$

> Use $\frac{\text{speed of rebound}}{\text{speed of approach}} = e$ with $e = e'$.

Using $e = \frac{5}{9}$, $w = \frac{u(5 + 1)}{4} = \frac{3u}{2}$

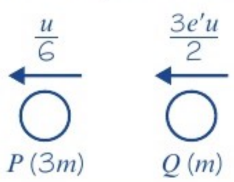
So $V = \frac{3e'u}{2}$

> Use your answer to part **a** to find w and use Newton's law of restitution to find V.

Also when $e = \frac{5}{9}$, $v = -\frac{u}{6}$

> Evaluate v using your answer to part **a** and $e = \frac{5}{9}$

After impact between Q and the wall



> Another diagram is helpful here.

Since Q and P collide again

$\frac{3e'u}{2} > \frac{u}{6}$

So $e' > \frac{1}{9}$

Problem-solving

If the two particles collide again, then the speed of Q must be greater than the speed of P.

Example 13

A tennis ball, which may be modelled as a particle, is dropped from rest at a height of 90 cm onto a smooth horizontal plane. The coefficient of restitution between the ball and the plane is 0.5. Assume that there is no air resistance and that the ball falls under gravity and hits the plane at right angles.

a Find the height to which the ball rebounds after the first bounce.

b Find the height to which the ball rebounds after the second bounce.

c Find the total distance travelled by the ball before it comes to rest, according to this model.

d Criticise this model with respect to the motion of the ball as it continues to bounce.

a As the tennis ball falls:

Use $v^2 = u^2 + 2as$

with $u = 0$, $s = 0.9$ and $a = g$

$v^2 = 1.8g$

$v = 4.2$

After first impact with plane:

By Newton's law of restitution,

$e = \frac{\text{speed of rebound}}{\text{speed of approach}}$

$0.5 = \frac{v'}{4.2}$

$v' = 2.1$

As the ball moves under gravity after impact:

Use $v^2 = u^2 + 2as$

with $v = 0$, $u = 2.1$ and $a = -g$

$0 = 2.1^2 - 2gh_1$

$h_1 = \frac{2.1^2}{2g}$

$= 0.225$

The ball rebounds after the first bounce to a height of 22.5 cm.

b As the tennis ball falls:

$v = v' = 2.1$

After second impact:

By Newton's law of restitution

$v'' = e \times 2.1 = 1.05$

As the ball moves under gravity after the second impact:

Use $v^2 = u^2 + 2as$

with $v = 0$, $u = 1.05$ and $a = -g$

Online Explore successive impacts of a falling particle using GeoGebra.

The tennis ball is falling under gravity so use the appropriate constant acceleration formula to find the ball's speed when it hits the plane.

v m s^{-1} is the approach speed.

v' m s^{-1} is the rebound speed of the particle.

After it rebounds it moves up under gravity. As the upward direction is taken as positive here, the acceleration is $-g$. Let $s = h_1$ when $v = 0$.

From symmetry v' m s^{-1} is the speed of the particle on its approach to the plane the second time.

Let v'' be the speed of the ball after the second bounce.

After it rebounds it moves under gravity to a height h_2. Again the acceleration is $-g$.

$0 = 1.05^2 - 2gh_2$

$h_2 = \dfrac{1.05^2}{2g} = 0.056\,25$

The ball rebounds after the second bounce to a height of 5.625 cm.

c The total distance travelled is

$0.9 + 0.225 + 0.225 + 0.056\,25 + 0.056\,25 + \dots$

$= 0.9 + 2(0.225 + 0.225 \times \frac{1}{4} + 0.225 \times (\frac{1}{4})^2 + 0.225 \times (\frac{1}{4})^3 + \dots)$

$= 0.9 + 2 \times \dfrac{0.225}{(1 - \frac{1}{4})} = 1.5$

The total distance travelled by the ball before it comes to rest is 1.5 m.

d According to this model the ball will bounce an infinite number of times. In real life, the ball will stop bouncing after a finite number of bounces.

Problem-solving

The ratio of the heights on each bounce is constant.
Initial height = 0.9 m, $h_1 = 0.9 \times \frac{1}{4} = 0.225$,
$h_2 = 0.9 \times (\frac{1}{4})^2 = 0.056\,25$, and so on.

The ball moves 0.9 m before first impact, then moves up 0.225 m and down 0.225 m before second impact then moves up 0.056 25 m and down 0.056 25 m before third impact.

Use the formula for the sum of an infinite geometric series:
$S = \dfrac{a}{1 - r}$ with $a = 0.225$ and $r = \frac{1}{4}$

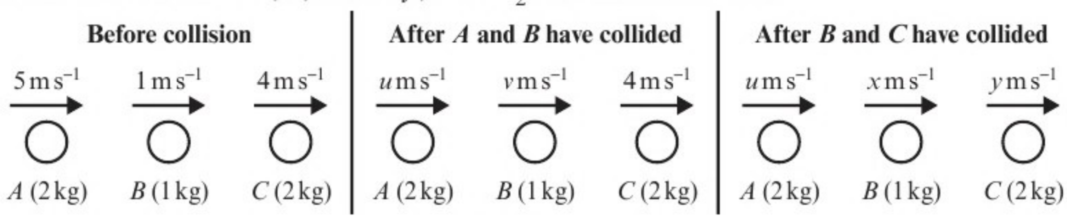
← Pure Year 2, Section 3.5

Exercise 4D

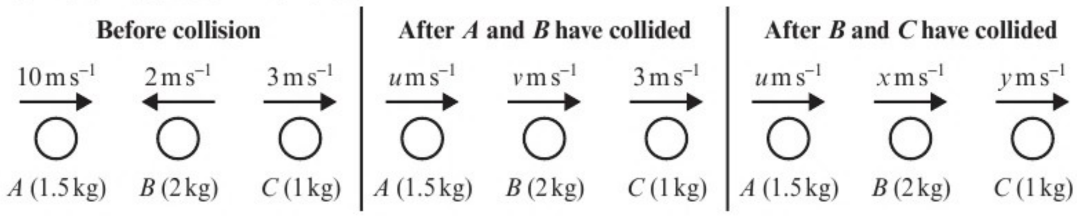
Whenever a numerical value of g is required, take $g = 9.8\,\text{m s}^{-2}$.

1 Three small smooth spheres A, B and C move along the same straight line on a horizontal plane. Sphere A collides with sphere B and then sphere B collides with sphere C. The diagrams show the velocities before the first collision, after the first collision between A and B and then after the collision between B and C.

a Find the values of u, v, x and y, if $e = \frac{1}{2}$ for both collisions.



b Find the values of u, v, x and y, if $e = \frac{1}{6}$ for the collision between A and B and $e = \frac{1}{2}$ for the collision between B and C.



(P) 2 Three perfectly elastic particles A, B and C of masses $3m$, $5m$ and $4m$ respectively lie at rest on a straight line on a smooth horizontal table with B between A and C. Particle A is projected directly towards B with speed $6\,\text{m s}^{-1}$ and after A has collided with B, B then collides with C. Find the velocity of each particle after the second impact.

Hint Perfectly elastic means $e = 1$.

(P) **3** Three identical smooth spheres A, B and C, each of mass m, lie at rest on a straight line on a smooth horizontal table. Sphere A is projected with speed u to strike sphere B directly. Sphere B then strikes sphere C directly. The coefficient of restitution between any two spheres is e, $e \neq 1$.

a Find the speeds in terms of u and e of the spheres after these two collisions.

b Show that A will catch up with B and there will be a further collision.

(E/P) **4** Three identical spheres A, B and C of equal mass m move along the same straight line on a horizontal plane. A and B are moving in opposite directions towards each other with speeds $4u$ and $2u$ respectively. C is moving in the same direction as A with speed $3u$.

a If the coefficient of restitution between any two of the spheres is e, show that B will only collide with C if $e > \frac{2}{3}$ **(8 marks)**

b Find the direction of motion of A after collision, if $e > \frac{2}{3}$ **(2 marks)**

(E/P) **5** Two particles P and Q of masses $2m$ and $3m$ respectively are moving in opposite directions on a smooth plane with speeds $4u$ and $2u$ respectively. The particles collide directly. The direction of motion of Q is reversed by the impact and its speed after impact is u. This particle then hits a smooth vertical wall perpendicular to its direction of motion. The coefficient of restitution between Q and the wall is $\frac{2}{3}$. In the subsequent motion, there is a further collision between Q and P. Find the velocities of P and Q after this collision. **(8 marks)**

(E/P) **6** Two small smooth spheres P and Q have masses m and $3m$ respectively. Sphere P is moving with speed $12u$ on a smooth horizontal table when it collides directly with Q which is at rest on the table. The coefficient of restitution between P and Q is $\frac{2}{3}$

a Find the velocities of P and Q immediately after the collision. **(6 marks)**

After the collision Q hits a smooth vertical wall perpendicular to the direction of its motion. The coefficient of restitution between Q and the wall is $\frac{4}{5}$. Q then collides with P a second time.

b Find the velocities of P and Q after the second collision between P and Q. **(8 marks)**

(E/P) **7** A small table tennis ball, which may be modelled as a particle, falls from rest at a height 40 cm onto a smooth horizontal plane. The coefficient of restitution between the ball and the plane is 0.7.

a Find the height to which the ball rebounds after:

i the first bounce

ii the second bounce. **(8 marks)**

b Describe the subsequent motion of the ball. **(1 mark)**

c Find the total distance travelled by the ball before it comes to rest. **(5 marks)**

Problem-solving

Find the ratio between the heights of successive bounces according to the model.

d Give one reason why your answer to part **c** is unrealistic. **(1 mark)**

E/P **8** A small smooth ball, which may be modelled as a particle, falls from rest at a height H onto a smooth horizontal plane. The coefficient of restitution between the ball and the plane is e.

a Find, in terms of H and e, the height to which the ball rebounds after the first bounce. **(9 marks)**

b Find, in terms of H and e, the height to which the ball rebounds after the second bounce. **(6 marks)**

c Find an expression for the total distance travelled by the ball before it comes to rest. **(6 marks)**

Problem-solving

Draw clear diagrams for each stage of the motion of the ball.

E/P **9** A ball B lies on a smooth horizontal plane between two smooth, parallel, vertical walls W_1 and W_2 that are d m apart. B is initially halfway between the walls and is projected with velocity $2\,\text{m s}^{-1}$ towards W_2. The coefficient of restitution between B and W_2 is e_2.

The ball then travels towards and strikes W_1. The coefficient of restitution between B and W_1 is e_1.

The ball then travels towards W_2 again.

Find, in terms of d, e_1 and e_2, the total time elapsed from the moment the ball is projected to the time when it strikes W_2 for the second time. **(9 marks)**

Challenge

Two particles P and Q, of equal mass, lie on a smooth horizontal plane between two smooth parallel, vertical walls W_1 and W_2 that are 4 m apart. P and Q are projected towards each other with speeds $2\,\text{m s}^{-1}$ and $1\,\text{m s}^{-1}$ respectively in such a way that they collide directly at a point equidistant from W_1 and W_2. The coefficient of restitution between the particles is 0.5. P then travels towards W_1, and Q travels towards W_2 and strikes W_2 perpendicularly. The coefficient of restitution between Q and W_2 is 0.4. Show that in the subsequent motion, P strikes W_1 before Q collides with P for a second time.

Mixed exercise 4

1 Two identical spheres, moving in opposite directions, collide directly. As a result of the impact one of the spheres is brought to rest. The coefficient of restitution between the spheres is $\frac{1}{3}$. Show that the ratio of the speeds of the spheres before the impact is 2 : 1.

2 A particle P of mass m is moving in a straight line with speed $\frac{1}{4}u$ at the instant when it collides directly with a particle Q, of mass λm, which is at rest. The coefficient of restitution between P and Q is $\frac{1}{4}$. Given that P comes to rest immediately after hitting Q find the value of λ.

E/P **3** A boy of mass m dives off a boat of mass M which was previously at rest. Immediately after diving off, the boy has horizontal speed v and the boat has horizontal speed V.

a Modelling the boy and the boat as particles in free space, find an expression for V in terms of v. **(4 marks)**

b Prove that the total kinetic energy of the boy and the boat is $\dfrac{m(m+M)v^2}{2M}$ **(4 marks)**

c Criticise the model in respect of the boat. **(1 mark)**

4 Two spheres P and Q of masses 4 kg and 2 kg respectively are travelling towards each other in opposite directions along a straight line on a smooth horizontal surface. Initially, P has a speed of $5\,\text{m s}^{-1}$ and Q has a speed of $3\,\text{m s}^{-1}$. After the collision the direction of Q is reversed and it is travelling at a speed of $2\,\text{m s}^{-1}$. Find the velocity of P after the collision and the loss of kinetic energy due to the collision.

E/P **5** A particle P of mass $3m$ is moving in a straight line with speed u at the instant when it collides directly with a particle Q of mass m which is at rest. The coefficient of restitution between P and Q is e.

a Show that after the collision P is moving with speed $\dfrac{u(3-e)}{4}$ **(6 marks)**

b Show that the loss of kinetic energy due to the collision is $\dfrac{3mu^2(1-e^2)}{8}$ **(10 marks)**

c Find, in terms of m, u and e, the impulse exerted on Q by P in the collision. **(4 marks)**

E/P **6** Two spheres of masses 70 g and 100 g respectively are moving in opposite directions with speeds $4\,\text{m s}^{-1}$ and $8\,\text{m s}^{-1}$ respectively. The spheres collide directly. The coefficient of restitution between the spheres is $\frac{5}{12}$. Find:

a the velocities of the spheres after impact **(4 marks)**

b the amount of kinetic energy lost in the collision. **(3 marks)**

E/P **7** A sphere of mass 2 kg moving at $35\,\text{m s}^{-1}$ catches up and collides directly with a sphere of mass 10 kg moving in the same direction at $20\,\text{m s}^{-1}$. Five seconds after the impact the 10 kg sphere encounters a fixed barrier which reduces it to rest. Assuming the coefficient of restitution between the spheres is $\frac{3}{5}$, find the time that will elapse before the 2 kg sphere strikes the 10 kg sphere again. You may assume that the spheres are moving on a smooth surface and have constant speed between collisions. **(10 marks)**

E/P **8** Three balls A, B and C of masses $4m$, $3m$ and $3m$ respectively lie at rest on a smooth horizontal table with their centres in a straight line. The coefficient of restitution between any pair of balls is $\frac{3}{4}$. Show that if A is projected towards B with speed V there are three impacts and the final velocities are $\frac{5}{32}V$, $\frac{1}{4}V$ and $\frac{7}{8}V$ respectively. **(12 marks)**

E **9** A bullet of mass 60 g is fired horizontally at a fixed vertical metal barrier. The bullet hits the barrier when it is travelling at $600\,\text{m s}^{-1}$ and then rebounds.

a Find the kinetic energy lost at the impact if $e = 0.4$. **(8 marks)**

b Give one possible form of energy into which the lost kinetic energy has been transformed. **(1 mark)**

E **10** A particle A of mass $4m$ moving with speed u on a horizontal plane strikes directly a particle B of mass $3m$ which is at rest on the plane. The coefficient of restitution between A and B is e.

a Find, in terms of e and u, the speeds of A and B immediately after the collision. **(8 marks)**

b Given that the magnitude of the impulse exerted by A on B is $2mu$, show that $e = \frac{1}{6}$ **(6 marks)**

(E/P) **11** A ball of mass m moving with speed kV on a smooth table catches up and collides with another ball of mass λm moving with speed V travelling in the same direction on the table. The impact reduces the first ball to rest.

a Show that the coefficient of restitution is $\dfrac{\lambda + k}{\lambda(k-1)}$ **(6 marks)**

b Show further that $\lambda > \dfrac{k}{k-2}$ and $k > 2$. **(6 marks)**

(E) **12** A ball is dropped from zero velocity and after falling for 1 s under gravity meets another identical ball which is moving upwards at $7\,\text{m s}^{-1}$.

a Taking the value of g as $9.8\,\text{m s}^{-2}$, calculate the velocity of each ball immediately after the impact, given that the coefficient of restitution is $\frac{1}{4}$ **(8 marks)**

b Find the percentage loss in kinetic energy due to the impact, giving your answer to 2 significant figures. **(4 marks)**

(E) **13** A particle falls from a height 8 m onto a fixed horizontal plane. The coefficient of restitution between the particle and the plane is $\frac{1}{4}$

a Find the height to which the particle rises after impact. **(6 marks)**

b Find the time the particle takes from leaving the plane after impact to reach the plane again. **(2 marks)**

c Find the speed of the particle after the second rebound. **(2 marks)**

You may leave your answers in terms of g.

(E/P) **14** A particle falls from a height h onto a fixed horizontal plane. If e is the coefficient of restitution between the particle and the plane, show that the total time taken before the particle finishes bouncing is $\dfrac{1+e}{1-e}\sqrt{\dfrac{2h}{g}}$ **(10 marks)**

(E/P) **15** A sphere P of mass m lies on a smooth table between a sphere Q of mass $8m$ and a fixed vertical plane. Sphere Q is initially at rest. Sphere P is projected towards sphere Q so they impact directly. The coefficient of restitution between the two spheres is $\frac{7}{8}$. Given that sphere P is reduced to rest by a second impact with sphere Q, find the coefficient of restitution between sphere P and the fixed vertical plane. **(12 marks)**

(E/P) **16** A gun of mass M kg is free to move horizontally. The gun fires a shell of mass m kg in a horizontal direction. The energy released by the explosion, which occurred in firing the shell, is E J. Find the velocity of the shell in terms of m, M and E if all of this energy is given to the shell and the gun. **(8 marks)**

(E/P) **17** A snooker ball of mass m kg is dropped from a point H m above a horizontal floor. The ball falls freely under gravity, strikes the floor and bounces to a height of h m. Given that the coefficient of restitution between the ball and the floor is e,

a show that $e = \sqrt{\dfrac{h}{H}}$ **(6 marks)**

b find the height the ball reaches after it bounces a second time **(3 marks)**

c describe the subsequent motion of the ball according to this model. **(2 marks)**

E/P **18** A small sphere B of mass 2 kg is held at rest 2 m up a smooth slope that is angled at 30° to the horizontal. When sphere B is released from rest it rolls down the slope onto a smooth, horizontal plane where it collides directly with a stationary small sphere C of mass 1 kg. Given that the coefficient of restitution between the balls is 0.75, calculate:

a the speed and direction of motion of B and C after the collision **(12 marks)**

b the kinetic energy lost in the collision. **(5 marks)**

c Without doing any further calculations, state how the amount of kinetic energy lost in the collision would change if $e < 0.75$. **(1 mark)**

Problem-solving

Resolve to find the acceleration of B.

E/P **19** Two spheres A and B sit on a smooth horizontal plane at a point P between two parallel, vertical walls W_1 and W_2 such that the ratio of distance $W_1P : PW_2$ is 2 : 1. Sphere A is projected towards W_1 with speed 2 m s^{-1} and B is projected towards W_2 with speed 3 m s^{-1}. The spheres rebound off their respective walls before colliding at a point Q. Given that the coefficient of restitution between both walls and spheres is 0.6, calculate the ratio of the distances $W_1Q : W_2Q$. **(8 marks)**

Challenge

Three small spheres A, B and C of masses m_1, m_2 and m_3 respectively lie in order in a straight line on a large smooth table. Two slack strings connect A to B and B to C. Sphere C is projected with speed u m s^{-1} away from the other two spheres. Show that the total kinetic energy of A, B and C when both strings are taut is

$$\frac{m_3^2u^2}{2(m_1 + m_2 + m_3)}$$

Summary of key points

1 **Newton's law of restitution** states that

$$\frac{\text{speed of separation of particles}}{\text{speed of approach of particles}} = e$$

The constant e is the **coefficient of restitution** between the particles $0 \leqslant e \leqslant 1$

2 For the direct collision of a particle with a smooth plane, Newton's law of restitution can be written as

$$\frac{\text{speed of rebound}}{\text{speed of approach}} = e$$

3 The loss of kinetic energy due to impact is

$$\left(\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2\right) - \left(\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2\right)$$

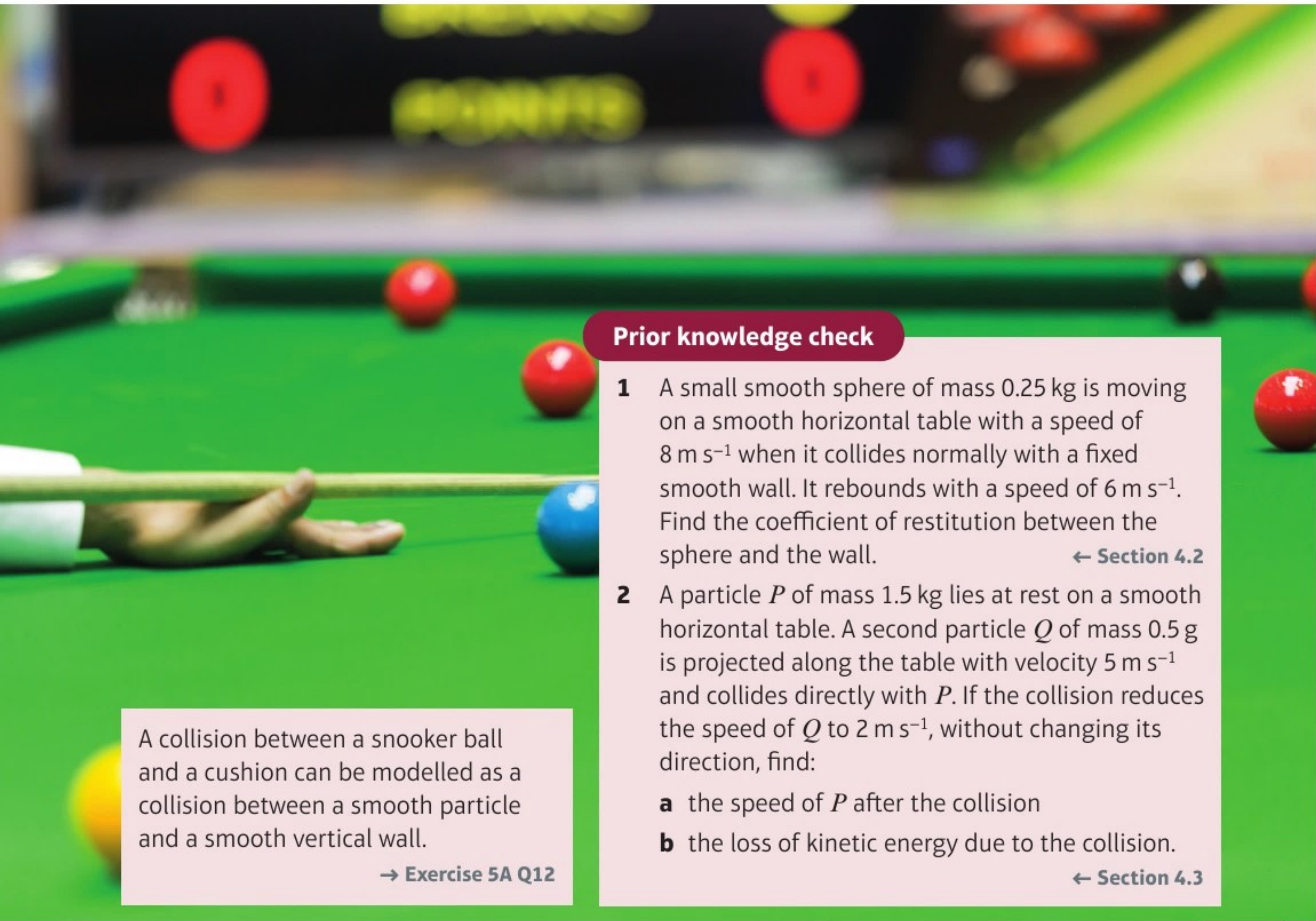
Elastic collisions in two dimensions

5

Objectives

After completing this chapter you should be able to:

- Solve problems involving the oblique impact of a smooth sphere with a fixed surface → pages 96–101
- Solve problems involving the oblique impact of two smooth spheres → pages 102–108
- Solve problems involving successive oblique impacts of a sphere with smooth plane surfaces → pages 108–115



A collision between a snooker ball and a cushion can be modelled as a collision between a smooth particle and a smooth vertical wall.

→ Exercise 5A Q12

Prior knowledge check

1. A small smooth sphere of mass 0.25 kg is moving on a smooth horizontal table with a speed of 8 m s^{-1} when it collides normally with a fixed smooth wall. It rebounds with a speed of 6 m s^{-1}. Find the coefficient of restitution between the sphere and the wall. ← Section 4.2
2. A particle P of mass 1.5 kg lies at rest on a smooth horizontal table. A second particle Q of mass 0.5 g is projected along the table with velocity 5 m s^{-1} and collides directly with P. If the collision reduces the speed of Q to 2 m s^{-1}, without changing its direction, find:

   a the speed of P after the collision

   b the loss of kinetic energy due to the collision.

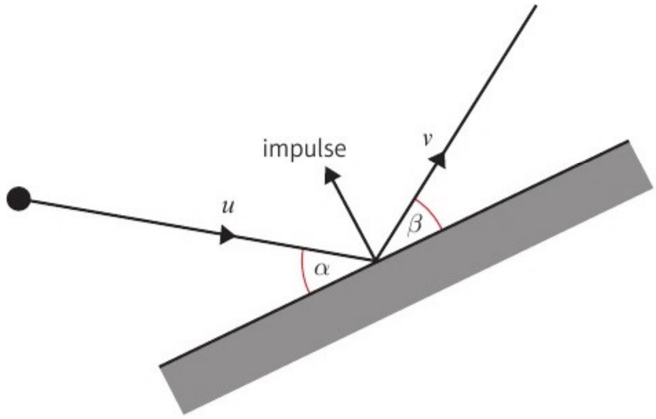
← Section 4.3

5.1 Oblique impact with a fixed surface

A You can solve problems involving the oblique impact of a smooth sphere with a smooth fixed surface.

- **When a smooth sphere collides with a smooth flat surface and bounces off it the velocity of the sphere changes, and therefore the momentum changes. The change in momentum is caused by the impact between the sphere and the surface.**

Notation An **oblique** impact is one where the particle does not strike a surface normally. You usually use α and β to represent the angles between the path of the particle and the surface before and after the impact.



Because the sphere and the surface are smooth, we know that the reaction between the sphere and the surface acts along the common normal at the point of contact. This means that the impulse must act in the same direction. Consequently, in the impact between a smooth sphere and a smooth fixed surface:

- **The impulse on the sphere acts perpendicular to the surface, through the centre of the sphere.**

You can solve problems involving oblique impacts with fixed surfaces by considering the components of the velocity of the sphere parallel and perpendicular to the surface.

- **The component of the velocity of the sphere parallel to the surface is unchanged.**

$$v \cos \beta = u \cos \alpha$$

- **You can use Newton's law of restitution to find the component of the velocity of the sphere perpendicular to the surface.**

$$v \sin \beta = eu \sin \alpha$$

You can eliminate u and v from the above relationships by dividing to obtain:

$$\frac{v \sin \beta}{v \cos \beta} = \frac{eu \sin \alpha}{u \cos \alpha}$$

$$\tan \beta = e \tan \alpha$$

And since $0 \leqslant e \leqslant 1$, $\tan \beta \leqslant \tan \alpha$, so $\beta \leqslant \alpha$.

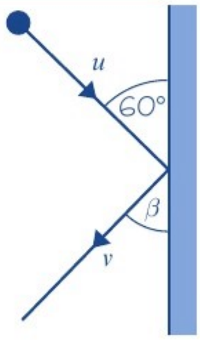
Notation The total angle through which the path of the sphere changes is sometimes called the **angle of deflection**. Using the above notation, then angle of deflection is $\alpha + \beta$.

Example 1

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of 60° with the wall. The coefficient of restitution between S and the wall is $\frac{1}{4}$. Find:

a the speed of S immediately after the collision

b the angle of deflection of S.



a For motion parallel to the surface:

$v\cos\beta = u\cos 60°$

$v\cos\beta = \frac{1}{2}u$ (1)

For motion perpendicular to the surface:

$v\sin\beta = eu\sin 60°$

$v\sin\beta = \frac{\sqrt{3}}{8}u$ (2)

Eliminating β from (1) and (2):

$v^2\sin^2\beta + v^2\cos^2\beta = \frac{3}{64}u^2 + \frac{1}{4}u^2$

$v^2(\sin^2\beta + \cos^2\beta) = \frac{19}{64}u^2$

$v = \frac{\sqrt{19}}{8}u$

b Eliminating v and u from (1) and (2):

$\frac{v\sin\beta}{v\cos\beta} = \frac{\frac{\sqrt{3}}{8}u}{\frac{1}{2}u}$

$\tan\beta = \frac{\sqrt{3}}{4}$

$\beta = 23.4°$ (1 d.p.)

Angle of deflection = 60° + 23.4° = 83.4° (1 d.p.)

Online Explore oblique impact with a fixed surface using GeoGebra.

Start with a diagram to show what is happening. In this chapter, you will only consider particles moving in a flat, horizontal plane impacting with flat, vertical surfaces.

The component of the velocity of S parallel to the surface is unchanged.

Use Newton's law of restitution for the component of the velocity perpendicular to the surface.

Solve (1) and (2) simultaneously. You can use $\sin^2\beta + \cos^2\beta \equiv 1$ to eliminate β.

Divide equation (2) by equation (1) to eliminate v and u. You could also substitute your value for v into one of the equations to find β.

Problem-solving

Both v and u are eliminated from the equations when you divide. The angle of deflection is **independent** of the original speed of the sphere.

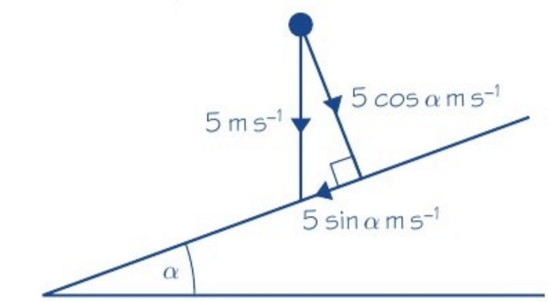
The same method can be used if the impact is with an inclined plane.

Example 2

A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{1}{2}$. Immediately before striking the plane the ball has speed $5\,\text{m s}^{-1}$. The coefficient of restitution between the ball and the plane is $\frac{1}{2}$

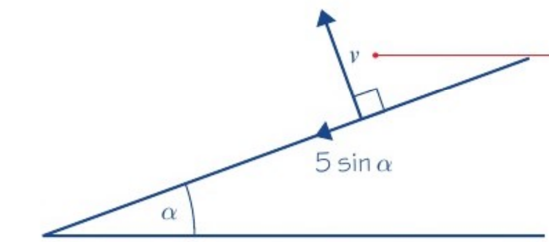
Find the speed of the ball immediately after the impact.

Before the impact:



You will need to consider the components of the velocity parallel and perpendicular to the surface separately, so it is useful to show them on your initial diagram.

After the impact:



Let v m s^{-1} be the component of the velocity of the ball after the impact that acts perpendicular to the surface.

The component of velocity parallel to the slope is $5\sin\alpha = 5 \times \frac{1}{\sqrt{5}} = \sqrt{5}$

There is no change in the component parallel to the slope.

Perpendicular to the slope:

$$v = e \times 5\cos\alpha = \tfrac{1}{2} \times 5 \times \frac{2}{\sqrt{5}} = \sqrt{5}$$

Newton's law of restitution applies perpendicular to the slope.

Therefore speed after impact

$$= \sqrt{\sqrt{5}^2 + \sqrt{5}^2} = \sqrt{10}\ \text{m s}^{-1}$$

The speed of the ball is the magnitude of the velocity vector.

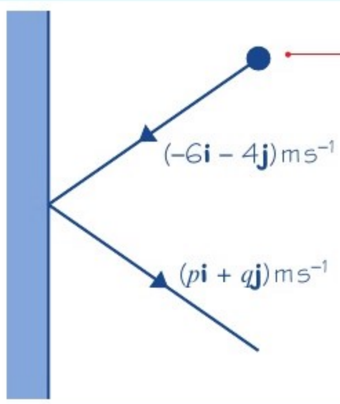
Example 3

A small smooth ball of mass 2 kg is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the y-axis. The velocity of the ball just before impact is $(-6\mathbf{i} - 4\mathbf{j})$ m s^{-1}. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$. Find:

a the velocity of the ball immediately after the impact

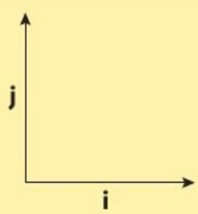
b the kinetic energy lost as a result of the impact

c the angle of deflection of the ball.



Start with a diagram.

Remember, **i** is a unit vector parallel to the x-axis, and **j** is a unit vector parallel to the y-axis.



A

a Let the velocity of the ball after the impact be $(p\mathbf{i} + q\mathbf{j})\,\text{m s}^{-1}$.
Parallel to the wall: $q = -4$
Perpendicular to the wall:
$p = \frac{1}{3} \times 6 = 2$
Velocity of the ball after impact
$= (2\mathbf{i} - 4\mathbf{j})\,\text{m s}^{-1}$

b Speed of ball before impact $= \sqrt{52}\,\text{m s}^{-1}$
K.E. of ball before impact $= \frac{1}{2} \times 2 \times (\sqrt{52})^2$
$= 52\,\text{J}$
Speed of ball after impact $= \sqrt{20}\,\text{m s}^{-1}$
K.E. of ball after impact $= \frac{1}{2} \times 2 \times (\sqrt{20})^2$
$= 20\,\text{J}$
Loss of K.E. $= 52 - 20 = 32\,\text{J}$

c Let θ be the angle of deflection:
$$\cos\theta = \frac{(-6)(2) + (-4)(-4)}{\sqrt{52}\sqrt{20}} = \frac{1}{\sqrt{65}}$$
$\theta = 82.9°$ (1 d.p.)

The initial velocity is already in component form with one component parallel to the wall and the other perpendicular to it.

Speed = $|\mathbf{v}|$ so use Pythagoras: $\sqrt{6^2 + 4^2} = \sqrt{52}$

K.E. $= \frac{1}{2}mv^2$

Problem-solving

You can use the **scalar product** to quickly determine the angle of deflection, θ. If the velocities before and after impact are $\mathbf{u}$ and $\mathbf{v}$ respectively, then $\cos\theta = \frac{\mathbf{u}.\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

← Core Pure Book 1, Section 9.3

Exercise 5A

1 A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of α with the wall, where $\tan\alpha = \frac{3}{4}$. The coefficient of restitution between S and the wall is $\frac{1}{3}$. Find:

a the speed of S immediately after the collision

b the angle of deflection of S.

2 A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of 30° with the wall. Immediately after the collision the speed of S is $\frac{7}{8}u$.
Find the coefficient of restitution between S and the wall.

3 A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of α with the wall, where $\tan\alpha = \frac{5}{12}$. The coefficient of restitution between S and the wall is $\frac{3}{5}$
Find the speed of S immediately after the collision.

E **4** A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of α with the wall, where $\tan\alpha = 2$. Immediately after the collision the speed of S is $\frac{3}{4}u$.
Find the coefficient of restitution between S and the wall. **(4 marks)**

A **5** A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle of 30° to the horizontal. Immediately before striking the plane the ball has speed 8 m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{1}{4}$

Find the exact value of the speed of the ball immediately after the impact.

6 A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle of 20° to the horizontal. Immediately before striking the plane the ball has speed 10 m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{2}{5}$. Find the speed, to 3 significant figures, of the ball immediately after the impact.

E **7** A small smooth ball of mass 750 g is falling vertically. The ball strikes a smooth plane which is inclined at an angle of 45° to the horizontal. Immediately before striking the plane the ball has speed $5\sqrt{2}\text{ m s}^{-1}$. The coefficient of restitution between the ball and the plane is $\frac{1}{2}$. Find:

Hint $\mathbf{I} = m\mathbf{v} - m\mathbf{u}$ ← Section 1.3

a the speed, to 3 significant figures, of the ball immediately after the impact **(2 marks)**

b the magnitude of the impulse received by the ball as it strikes the plane. **(2 marks)**

E **8** A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. Immediately before striking the plane the ball has speed 7.5 m s^{-1}. Immediately after the impact the ball has speed 5 m s^{-1}.

Find the coefficient of restitution, to 2 significant figures, between the ball and the plane. **(4 marks)**

9 A small smooth ball of mass 800 g is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the y-axis. The velocity of the ball just before impact is $(5\mathbf{i} - 3\mathbf{j})\text{ m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{2}$. Find:

a the velocity of the ball immediately after the impact

b the kinetic energy lost as a result of the impact

c the angle of deflection of the ball.

10 A small smooth ball of mass 1 kg is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the x-axis. The velocity of the ball just before impact is $(3\mathbf{i} + 6\mathbf{j})\text{ m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$. Find:

a the speed of the ball immediately after the impact

b the kinetic energy lost as a result of the impact.

P **11** A small smooth ball of mass 2 kg is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the line $y = x$. The velocity of the ball just before impact is $(4\mathbf{i} + 2\mathbf{j})\text{ m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$. Find:

a the velocity of the ball immediately after the impact

b the proportion of the original kinetic energy lost as a result of the impact

c the angle of deflection of the ball.

12 A smooth snooker ball strikes a smooth cushion with speed 8 m s^{-1} at an angle of 45° to the cushion. Given that the coefficient of restitution between the ball and the cushion is $\frac{2}{5}$, find the magnitude and direction of the velocity of the ball after the impact.

A P **13** A smooth snooker ball strikes a smooth cushion with speed $u\,\text{m}\,\text{s}^{-1}$ at an angle of 50° to the cushion. The coefficient of restitution between the ball and the cushion is e.

a Show that the angle between the cushion and the rebound direction is independent of u.

b Find the value of e given that the ball rebounds at right angles to its original direction.

14 A smooth billiard ball strikes a smooth cushion at an angle of α to the cushion, where $\tan\alpha = \frac{3}{4}$ The ball rebounds at an angle of β to the cushion, where $\tan\beta = \frac{5}{12}$. Find:

a the fraction of the kinetic energy of the ball lost in the collision

b the coefficient of restitution between the ball and the cushion.

P **15** A small smooth sphere of mass m is moving with velocity $(5\mathbf{i} - 2\mathbf{j})\,\text{m}\,\text{s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(2\mathbf{i} + 2\mathbf{j})\,\text{m}\,\text{s}^{-1}$. Find:

a a unit vector in the direction of the impulse received by the sphere

b the coefficient of restitution between the sphere and the wall.

Problem-solving

Use $\mathbf{I} = m\mathbf{v} - m\mathbf{u}$ to find the direction of the impulse. The component of a velocity, $\mathbf{v}$, which acts in this direction is the scalar product $\mathbf{v}.\hat{\mathbf{I}}$, where $\hat{\mathbf{I}}$ is your answer to part **a**.

E/P **16** A small smooth sphere of mass 2 kg is moving with velocity $(2\mathbf{i} + 3\mathbf{j})\,\text{m}\,\text{s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(3\mathbf{i} - \mathbf{j})\,\text{m}\,\text{s}^{-1}$. Find:

a the magnitude and direction of the impulse received by the sphere **(3 marks)**

b the coefficient of restitution between the sphere and the wall **(5 marks)**

c the kinetic energy lost by the sphere in the collision. **(4 marks)**

E/P **17** A sphere slides across a smooth horizontal plane towards a smooth vertical wall with a speed of $3v$ and at an angle of α to the wall. After striking the wall the sphere moves at right-angles to its original direction of motion with speed v.

a Show that $\tan\alpha = 3$. **(3 marks)**

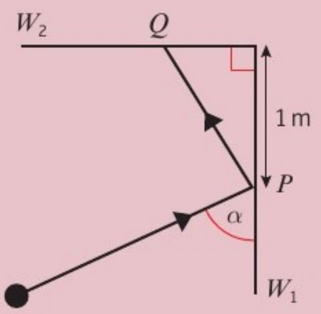
b Find the coefficient of restitution between the sphere and the wall. **(4 marks)**

E/P **18** A particle P is moving in a vertical plane when it collides with a smooth fixed horizontal wall. Immediately before the collision the angle between the direction of motion of P and the wall is α where $0° < \alpha < 90°$. Immediately after the collision, the angle between the direction of motion of P and the wall is $\frac{1}{2}\alpha$. Given that the coefficient of restitution between P and the wall is 0.4, find the value of α. **(10 marks)**

Challenge

A ball is moving on a smooth horizontal surface with speed u when it collides with a smooth fixed vertical wall W_1 at a point P. At the instant of the impact the ball is moving at an angle of α to the wall. The coefficient of restitution between the wall and the ball is e. The ball then moves on to collide with a second vertical wall W_2 at point Q. W_2 is perpendicular to W_1 and the perpendicular distance from point P to W_2 is 1 m.

Show that the distance $PQ = \sqrt{e^2\tan^2\alpha + 1}$.



5.2 Successive oblique impacts

A

You can solve problems involving successive oblique impacts of a sphere with smooth plane surfaces.

You should consider each impact separately, and draw separate diagrams for each one.

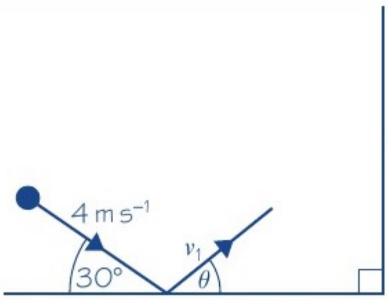
Example 4

Two vertical walls meet at right angles. A smooth sphere slides across a smooth, horizontal floor, bouncing off each wall in turn. Just before the first impact the sphere is moving with speed $4\,\text{m s}^{-1}$ at an angle of 30° to the wall. The coefficient of restitution between the sphere and both walls is $\frac{3}{4}$
Find:

a the direction of motion and speed of the sphere after the first collision

b the direction of motion and speed of the sphere after the second collision.

a



Draw a diagram for the first collision.

$v_1 \cos\theta = 4\cos 30° = 2\sqrt{3}$

The component of the ball's velocity parallel to the wall is unchanged.

$v_1 \sin\theta = 4 \times \frac{3}{4}\sin 30°$

Use Newton's law of restitution for motion perpendicular to the wall.

$v_1 \sin\theta = \frac{3}{2}$

$\frac{v_1 \sin\theta}{v_1 \cos\theta} = \frac{3}{4\sqrt{3}}$

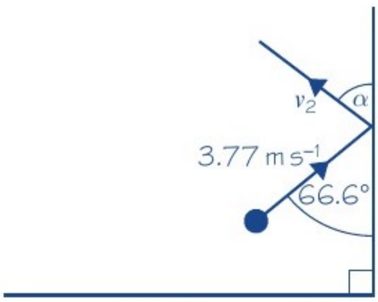
$\tan\theta = \frac{\sqrt{3}}{4}$

Eliminate v_1 and find $\tan\theta$.

so $\theta = 23.4°$ (3 s.f.) and $v_1 = 3.77\,\text{ms}^{-1}$ (3 s.f.)

Find v_1 and θ. Once you have found θ you can use substitution to find v_1, or you can use $\sin^2\theta + \cos^2\theta \equiv 1$ to eliminate θ.

b



Draw a new diagram for the second collision. The walls are perpendicular so the angle of impact will be 90° − 23.4° = 66.6°

$v_2 \cos\alpha = 3.774\ldots\cos 66.58\ldots° = 1.5$

The component of the velocity parallel to the wall is unchanged.

$$\frac{v_2 \sin \alpha}{3.774\ldots \sin 66.58\ldots°} = \frac{3}{4}$$

Use Newton's law of restitution for motion perpendicular to the wall.

$v_2 \sin \alpha = 2.598\ldots$

$$\frac{v_2 \sin \alpha}{v_2 \cos \alpha} = \frac{2.598\ldots}{1.5}$$

Eliminate v_2 and find $\tan \alpha$.

$\tan \alpha = 1.732\ldots$

so $\alpha = 60°$ and $v_2 = 3\,\text{m s}^{-1}$

Find v_2 and α.

So the sphere is travelling at $3\,\text{m s}^{-1}$ at an angle of 60° to the second wall.

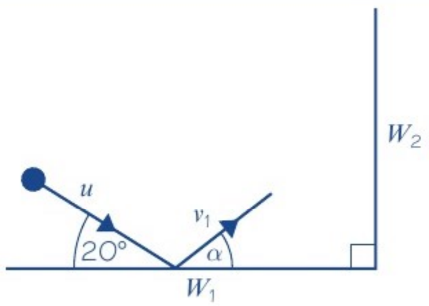
Example 5

Two cushions of a snooker table W_1 and W_2 meet at right angles. A snooker ball travels across the table and collides with W_1 and then W_2. The cushions are modelled as smooth.
Just before the first impact the ball is moving with speed $u\,\text{m s}^{-1}$ at an angle of 20° to W_1.
The coefficients of restitution between the ball and the cushions W_1 and W_2 are $\frac{1}{2}$ and $\frac{2}{5}$ respectively.

a Find the percentage of the ball's original kinetic energy that is lost in the collisions.

b In reality the cushions may not be smooth. What effect will the model have had on the calculation of the percentage of kinetic energy remaining?

a



Draw a diagram for the first collision.

$v_1 \cos \alpha = u \cos 20°$

$v_1 \sin \alpha = \frac{1}{2} u \sin 20°$

Use Newton's law of restitution for motion perpendicular to the cushion.

$$\frac{v_1 \sin \alpha}{v_1 \cos \alpha} = \frac{\frac{1}{2} u \sin 20°}{u \cos 20°}$$

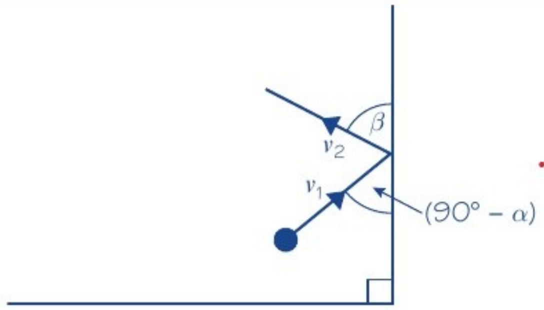
$\tan \alpha = \frac{1}{2} \tan 20°$

Eliminate v_1 and find $\tan \alpha$.

Problem-solving

You could find α, and v_1 in terms of u, but you do not necessarily need this information. Leave your expression for $\tan \alpha$ as it is for now, as you might be able to use it in this form later in the question.

A



Draw a new diagram for the second collision.

$v_2 \cos\beta = v_1 \cos(90° - \alpha) = v_1 \sin\alpha$

$v_2 \sin\beta = v_1 \times \frac{2}{5}\sin(90° - \alpha)$

$v_2 \sin\beta = \frac{2}{5}v_1 \cos\alpha$

$\frac{v_2 \sin\beta}{v_2 \cos\beta} = \frac{\frac{2}{5}v_1 \cos\alpha}{v_1 \sin\alpha}$

Watch out The coefficient of restitution between the ball and the second cushion is not the same as the coefficient between the ball and the first cushion.

$\tan\beta = \frac{2}{5\tan\alpha}$ but $\tan\alpha = \frac{1}{2}\tan 20°$

so $\tan\beta = \frac{4}{5\tan 20°}$

Eliminate v_2 and find $\tan\beta$.

$\beta = 65.5°$ (3 s.f.)

$v_2 \cos\beta = v_1 \sin\alpha$

$= \frac{1}{2}u \sin 20°$

You need to find an expression for v_2 in terms of u. Use the expression for $v_1 \sin\alpha$ you calculated earlier together with your unrounded value for β.

$v_2 = \frac{\frac{1}{2}u \sin 20°}{\cos 65.53...°}$

$= 0.4129...u$

K.E. before collisions $= \frac{1}{2}mu^2$

K.E. after collisions $= \frac{1}{2}m(0.4129...u)^2$

$= \frac{1}{2}mu^2 \times 0.1705...$

Therefore 83% of the ball's original kinetic energy was lost.

Final K.E. = 0.17 × original K.E.
This represents an 83% reduction.

b The calculation of the percentage of K.E. lost is actually too small. In reality, more K.E. will be lost as the rough surface will result in lower final speeds.

State how the answer will change and justify your statement.

Example 6

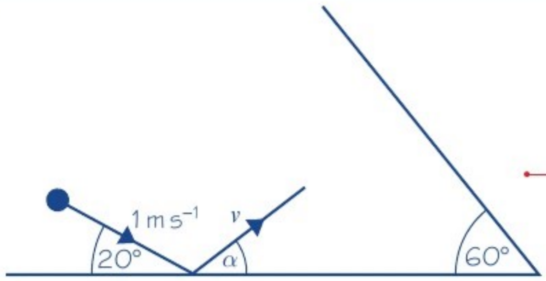
Two smooth vertical walls stand on a smooth horizontal surface and intersect at an angle of 60°. A smooth sphere is projected across the surface with speed $1\,\text{m}\,\text{s}^{-1}$ at an angle of 20° to one of the walls and towards the intersection of the walls. The coefficient of restitution between the sphere and the walls is 0.4. Work out the speed and direction of motion of the sphere after:

a the first collision **b** the second collision

Online Explore successive oblique impacts with a fixed surface using GeoGebra.

A

a



Draw a diagram showing both walls and the first collision.

$v\cos\alpha = \cos 20°$

$v\sin\alpha = 0.4\sin 20°$

Use Newton's law of restitution for motion perpendicular to the wall.

$\frac{v\sin\alpha}{v\cos\alpha} = \frac{0.4\sin 20°}{\cos 20°}$

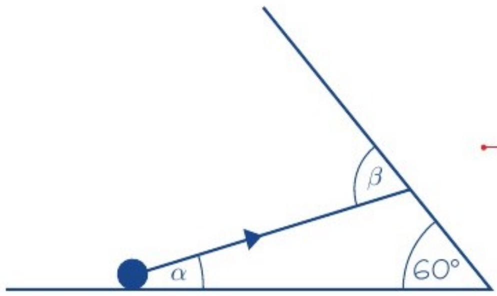
Eliminate v_1 and find $\tan\alpha$.

$\tan\alpha = 0.4\tan 20°$

so $\alpha = 8.28°$ (3 s.f.) and $v = 0.950$ (3 s.f.)

Find v and α.

b

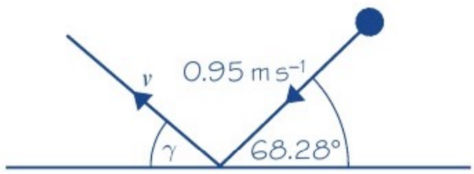


Draw a diagram and use angle properties of a triangle to work out the size of β.

$\beta = \alpha + 60° = 68.28...°$

Problem-solving

Draw a separate diagram for the second collision. You don't have to draw it in the same orientation: it can sometimes help to rotate the diagram so you can see what is going on more clearly.



$v\cos\gamma = 0.95\cos 68.28...°$

$v\sin\gamma = 0.4 \times 0.95\sin 68.28...°$

Use Newton's law of restitution for motion perpendicular to the wall.

$\frac{v\sin\gamma}{v\cos\gamma} = \frac{0.4 \times 0.95\sin 68.28...°}{0.95\cos 68.28...°}$

$\tan\gamma = 1.004...$

Eliminate v and find $\tan\gamma$.

$\gamma = 45.1°$ (3 s.f.) and $v = 0.498\text{ m s}^{-1}$ (3 s.f.)

Find v and γ.

Exercise 5B

A

1 Two smooth vertical walls stand on a smooth horizontal surface and intersect at right angles. A smooth sphere is moving across the surface and bounces off each wall in turn. At the moment just before the first impact the sphere has speed $2\,\mathrm{m\,s^{-1}}$ and is travelling at an angle of 30° to the first wall. The coefficient of restitution between the sphere and each wall is 0.5. Work out the speed and direction of motion of the sphere after:

a the first collision

b the second collision.

P

2 Two smooth vertical walls stand on a smooth horizontal surface and intersect at right angles. A smooth sphere is moving in the xy-plane such that it collides with the first wall at a speed of $1\,\mathrm{m\,s^{-1}}$ at an angle of 40° to the wall. The coefficient of restitution between the sphere and both walls is e. Given that after the first collision the sphere is moving at an angle of 20° to the wall, work out:

a the speed of the sphere after the first collision

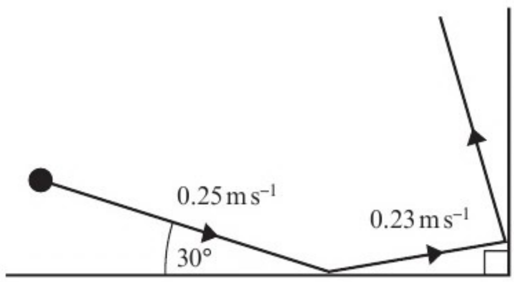
b the value of e.

The sphere then moves on to collide with the second wall.

c Calculate the speed and direction of motion of the sphere after the second collision.

E/P

3 Two smooth vertical walls stand on a smooth horizontal surface and intersect at right angles. A smooth sphere of mass 0.1 kg is moving in the xy-plane such that it collides with the first wall at a speed of $0.25\,\mathrm{m\,s^{-1}}$ at an angle of 30° to the wall. The coefficient of restitution between the sphere and both walls is e. Given that after the first collision the sphere is moving with speed $0.23\,\mathrm{m\,s^{-1}}$, work out:



a the direction in which the sphere is moving **(2 marks)**

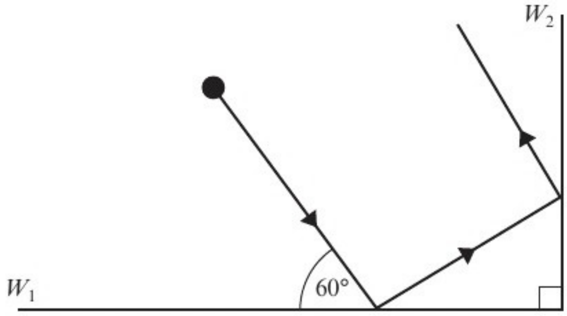
b the value of e. **(2 marks)**

The sphere then moves on to collide with the second wall.

c Calculate the kinetic energy of the sphere after the second collision. **(6 marks)**

E/P

4 Two smooth vertical walls, W_1 and W_2, stand on a smooth horizontal surface and intersect at right angles. A smooth ball of mass 2 kg is moving in the xy-plane such that it collides with the first wall at an angle of 60° to the wall. The coefficients of restitution between the ball and the walls W_1 and W_2 are 0.75 and 0.6 respectively. Given that immediately before the first collision the ball has kinetic energy of 9 J, work out:



a the speed and direction of motion of the ball after the first collision **(4 marks)**

b the kinetic energy of the ball after the second collision. **(4 marks)**

A P **5** Two vertical walls meet at right angles at the corner of a room. A small smooth sphere slides across the floor and bounces off each wall in turn. Just before the first impact the sphere is moving with speed $u\,\text{m s}^{-1}$ at an acute angle α to the wall. The coefficient of restitution between the sphere and wall is e. Find the speed and direction of motion of the sphere after the second collision.

P **6** Two smooth vertical walls stand on a smooth horizontal floor and intersect at an angle of 30°. A particle is projected along the floor with speed $u\,\text{m s}^{-1}$ at 45° to one of the walls and towards the intersection of the walls. The coefficient of restitution between the particle and each wall is $\frac{1}{\sqrt{3}}$

Find the speed of the particle after one impact with each wall.

E/P **7** Two vertical walls stand on a smooth horizontal surface and intersect at an angle of 45°. A smooth sphere is projected across the surface with speed $5\,\text{m s}^{-1}$ at an angle of 30° to the first wall and towards the intersection of the walls. The coefficient of restitution between the sphere and both walls is 0.8. Work out the speed and direction of motion of the sphere after:

a the first collision **(4 marks)**

b the second collision. **(4 marks)**

In reality the coefficients of restitution between the sphere and the first wall, e_1, and the sphere and the second wall, e_2, may not be the same.

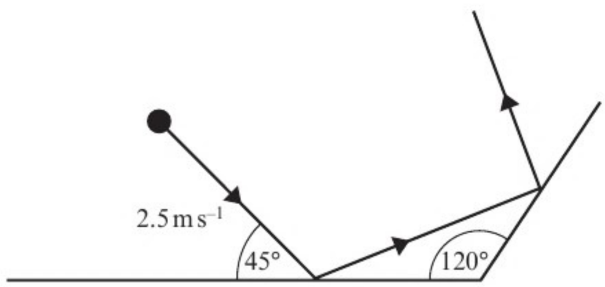
c Without doing any further calculation, state how the speed and direction of motion of the sphere after the second collision would change if $e_2 > 0.8$. **(1 mark)**

E/P **8** Two smooth vertical walls stand on a smooth horizontal surface and intersect at right angles, parallel with the vectors **i** and **j**. A small smooth sphere of mass m is moving with velocity $(5\mathbf{i} - 4\mathbf{j})\,\text{m s}^{-1}$ when it hits one of the walls. It rebounds from the wall with velocity $(5\mathbf{i} + 2\mathbf{j})\,\text{m s}^{-1}$ and goes on to hit the second wall. The coefficients of restitution between the sphere and each wall are the same. Find:

a the kinetic energy lost by the sphere in the collisions **(5 marks)**

b the total angle through which the sphere is deflected in its motion. **(3 marks)**

E/P **9** Two smooth vertical walls stand on a smooth horizontal surface and intersect at an angle of 120°. A smooth sphere of mass 0.1 kg is projected across the surface with speed $2.5\,\text{m s}^{-1}$ at an angle of 45° to one of the walls and towards the intersection of the walls. The coefficient of restitution between the sphere and the walls is 0.6.



a Work out the speed and direction of motion of the sphere after the first collision. **(6 marks)**

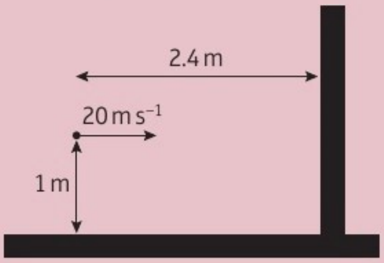
The sphere then moves on to collide with the second wall.

b Calculate the kinetic energy of the sphere after the second collision. **(8 marks)**

A E/P **10** Two vertical walls W_1 and W_2 stand on a smooth horizontal surface and intersect at an angle of 75°. A smooth sphere of mass 2 kg is projected across the surface with speed $10\,\text{m}\,\text{s}^{-1}$ at an angle of 30° to one of the walls and towards the intersection of the walls. The coefficient of restitution between the sphere and walls W_1 and W_2 is 0.7 and 0.5 respectively. Calculate the total kinetic energy lost by the sphere. **(10 marks)**

Challenge

A squash player hits a ball towards the wall of a squash court. The ball is modelled as a particle projected horizontally at a speed of $20\,\text{m}\,\text{s}^{-1}$. The point of projection is 1 m above the horizontal floor and 2.4 m from the vertical wall. The ball travels in a vertical plane perpendicular to the wall and floor. The coefficient of restitution between the ball and each surface is 0.6.



a Find:

i the distance of the ball from the wall when it bounces on the floor for the first time

ii the maximum height reached by the squash ball after this first bounce.

When a squash ball is heated up, the coefficient of restitution between the ball and a surface is increased.

b State how heating the squash ball would affect your answers to part **a**.

5.3 Oblique impact of smooth spheres

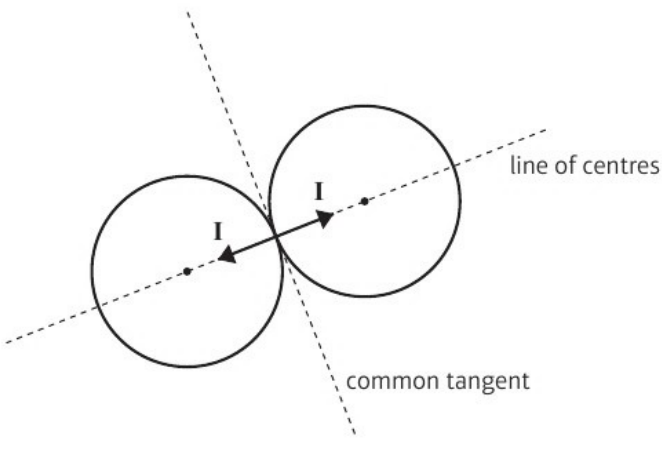
You can solve problems involving the oblique impact of two smooth spheres.

The methods used to solve problems involving the impact between a sphere and a fixed surface can be adapted to solve problems involving the impact of two spheres.

At the moment of impact the two spheres have a common tangent, which is perpendicular to the line through the centres of the two spheres.

- **The reaction between the two spheres acts along the line of centres, so the impulse affecting each sphere also acts along the line of centres.**

Notation An **oblique** impact between two spheres is one in which the two spheres are not travelling along the same straight line.



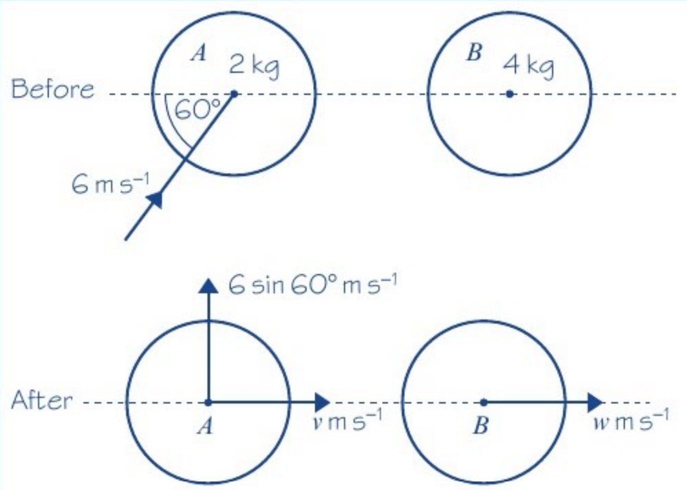
A

- **The components of the velocities of the spheres perpendicular to the line of centres are unchanged in the impact.**
- **Newton's law of restitution applies to the components of the velocities of the spheres parallel to the line of centres.**
- **The principle of conservation of momentum applies parallel to the line of centres.**

The principle of conservation of momentum also applies perpendicular to the line of centres, but as the components of velocity in this direction are unchanged you will not need to use this in your calculations.

Example 7

A smooth sphere A, of mass 2 kg and moving with speed $6\,\mathrm{m\,s^{-1}}$ collides obliquely with a smooth sphere B of mass 4 kg. Just before the impact B is stationary and the velocity of A makes an angle of 60° with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{4}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.



Problem-solving

Start by drawing 'before' and 'after' diagrams, and identify all the components that you need to find.

Let v and w represent the components of the velocities of A and B after the collision that act parallel to the line of centres.

Since B is stationary before the impact it will be moving along the line of centres after the impact.

For motion perpendicular to line of centres:

Component of velocity for $A = 6\sin 60° = 3\sqrt{3}$

No change in the component perpendicular to the line of centres.

For motion parallel to the line of centres:

$2 \times 6\cos 60° = 2v + 4w$

Conservation of momentum along the line of centres.

$6 = 2v + 4w \qquad (1)$

$w - v = \frac{1}{4} \times 6\cos 60°$

Newton's law of restitution.

$w - v = \frac{3}{4} \qquad (2)$

So $w = \frac{5}{4}$ and $v = \frac{1}{2}$

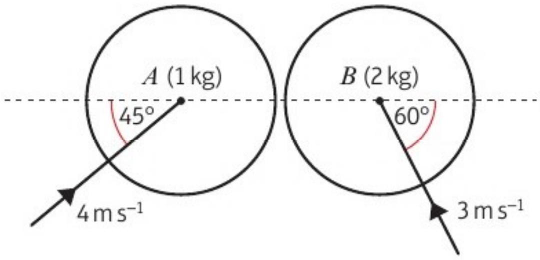
Solve equations (1) and (2) simultaneously.

A

Speed of A is $\sqrt{(3\sqrt{3})^2 + (\frac{1}{2})^2} = \sqrt{\frac{109}{4}}$ m s^{-1} at $\arctan\left(\frac{3\sqrt{3}}{\frac{1}{2}}\right) = 84.5°$ to the line of centres, and speed of B is $\frac{5}{4}$ m s^{-1} along the line of centres.

Combine the two components of the velocity of A.

Example 8

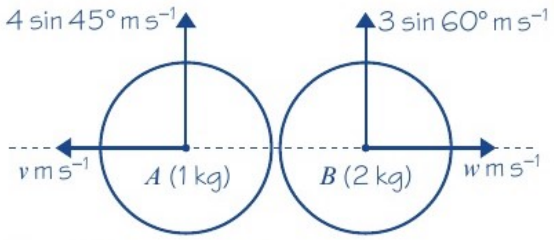


A small smooth sphere A of mass 1 kg collides with a small smooth sphere B of mass 2 kg. Just before the impact A is moving with a speed of 4 m s^{-1} in a direction at 45° to the line of centres and B is moving with speed 3 m s^{-1} at 60° to the line of centres, as shown in the diagram. The coefficient of restitution between the spheres is $\frac{3}{4}$. Find:

a the kinetic energy lost in the impact

b the magnitude of the impulse exerted by A on B.

a



You have been given a diagram of the situation before the impact, so draw a diagram showing the components of the velocities of each sphere after the impact. There is no change to the component of the velocities perpendicular to the line of centres, so include these in your diagram.

Parallel to the line of centres:

$1 \times 4\cos 45° - 2 \times 3\cos 60° = 2w - v$

Conservation of momentum.

$2\sqrt{2} - 3 = 2w - v \qquad (1)$

$v + w = \frac{3}{4}(4\cos 45° + 3\cos 60°)$

Newton's law of restitution.

$v + w = \frac{3\sqrt{2}}{2} + \frac{9}{8} \qquad (2)$

$3w = \frac{7\sqrt{2}}{2} - \frac{15}{8}$

Solve the simultaneous equations to find v and w.

$w = \frac{7\sqrt{2}}{6} - \frac{5}{8} = 1.0249\ldots$

$v = \frac{\sqrt{2}}{3} + \frac{7}{4} = 2.2214\ldots$

Total K.E. before impact

$= \frac{1}{2} \times 1 \times 4^2 + \frac{1}{2} \times 2 \times 3^2 = 17$ J

K.E. $= \frac{1}{2}mv^2$

A

Speed of A after impact $= \sqrt{(2\sqrt{2})^2 + 2.2214\ldots^2}$
$= 3.5964\ldots\text{ m s}^{-1}$

Speed of B after impact =
$\sqrt{\left(\frac{3\sqrt{3}}{2}\right)^2 + 1.0249\ldots^2} = 2.7929\ldots\text{ m s}^{-1}$

Total K.E. after impact $= \frac{1}{2} \times 1 \times 3.5964\ldots^2 + \frac{1}{2} \times 2 \times 2.7929\ldots^2 = 14.267\ldots\text{ J}$

Total loss of K.E. $= 17 - 14.3 = 2.73\text{ J}$ (3 s.f.)

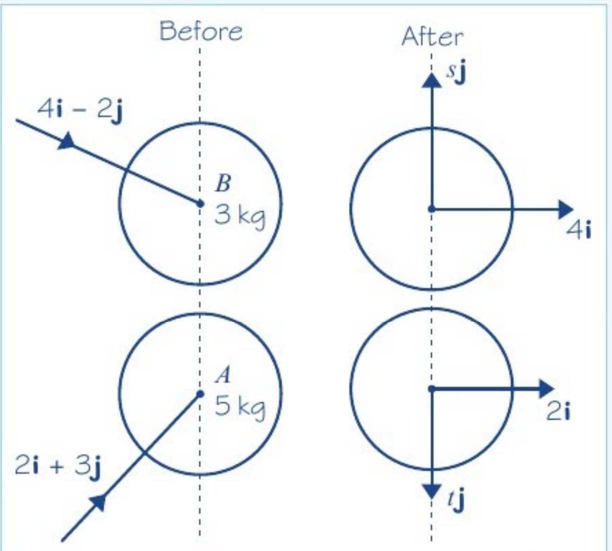
b Impulse = change in momentum
$= 2(1.0249\ldots - (-3\cos 60°))$
$= 5.05\text{ N s}$ (3 s.f.)

Watch out As energy is a scalar quantity the safest way to find the total loss (or the percentage loss) of kinetic energy is to consider the speed of each particle before and after the collision.

The component of the momentum that acts perpendicular to the line of centres is unchanged. Consider the change in momentum in either particle parallel to the line of centres.

Example 9

A smooth sphere A of mass 5 kg is moving on a smooth horizontal surface with velocity $(2\mathbf{i} + 3\mathbf{j})\text{ m s}^{-1}$. Another smooth sphere B of mass 3 kg and the same radius as A is moving on the same surface with velocity $(4\mathbf{i} - 2\mathbf{j})\text{ m s}^{-1}$. The spheres collide when their line of centres is parallel to $\mathbf{j}$. The coefficient of restitution between the spheres is $\frac{3}{5}$. Find the velocities of both spheres after the impact.



Online Explore oblique impacts of smooth spheres using GeoGebra.

Problem-solving The diagram helps you to understand the relative positions of A and B when they collide – if they were the other way round they would be moving apart.

There is no change in the components of velocity perpendicular to the line of centres.
Parallel to the line of centres:
$3 \times (-2) + 5 \times 3 = 3 \times s - 5 \times t$, — Conservation of momentum.
$9 = 3s - 5t$ (1)
$s + t = \frac{3}{5}(2 + 3)$, $s + t = 3$, — Newton's law of restitution.
$3s + 3t = 9$ (2)
$\Rightarrow t = 0$, $s = 3$ — Solve for s and t.
Velocity of A is $2\mathbf{i}\text{ m s}^{-1}$, and velocity of B is $(4\mathbf{i} + 3\mathbf{j})\text{ m s}^{-1}$.

Example 10

A

Two small smooth spheres A and B have equal radii. The mass of A is $2m$ kg and the mass of B is $3m$ kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $5\mathbf{j}$ m s^{-1} and the velocity of B is $(3\mathbf{i} - \mathbf{j})$ m s^{-1}. Immediately after the collision the velocity of A is $(3\mathbf{i} + 2\mathbf{j})$ m s^{-1}. Find:

a the speed of B immediately after the collision

b a unit vector parallel to the line of centres of the spheres at the instant of the collision.

a If the velocity of B after the collision is $\mathbf{v}$ then

$2m(5\mathbf{j}) + 3m(3\mathbf{i} - \mathbf{j}) = 2m(3\mathbf{i} + 2\mathbf{j}) + 3m\mathbf{v}$

$3\mathbf{v} = (9 - 6)\mathbf{i} + (10 - 3 - 4)\mathbf{j} = 3\mathbf{i} + 3\mathbf{j}$

$\mathbf{v} = (\mathbf{i} + \mathbf{j})$ m s^{-1}

Speed of $B = \sqrt{1^2 + 1^2} = \sqrt{2}$ m s^{-1}

b In the collision A receives an impulse of

$2m((3\mathbf{i} + 2\mathbf{j}) - 5\mathbf{j}) = 2m(3\mathbf{i} - 3\mathbf{j}) = 6m(\mathbf{i} - \mathbf{j})$ N s

$\Rightarrow$ the line of centres is parallel to the unit vector $\frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

Form a vector equation for conservation of momentum. ← Section 1.3

Speed = $|\mathbf{v}|$

The direction of the impulses on the spheres is parallel to the line of centres.

Watch out Note that in this example it is much simpler to form a single vector equation for conservation of momentum than to find the components of the velocities parallel to and perpendicular to the line of centres.

Exercise 5C

1 A small smooth sphere A, of mass 2 kg and moving with speed 6 m s^{-1} collides obliquely with a small smooth sphere B of mass 4 kg. Just before the impact B is stationary and the velocity of A makes an angle of 10° with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

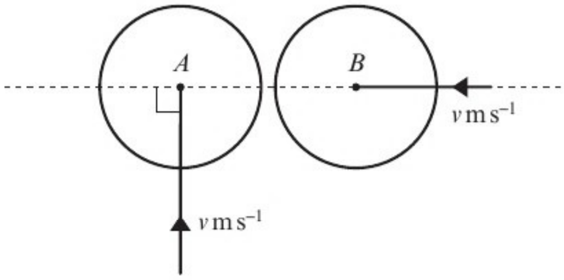
2 A smooth sphere A, of mass 4 kg and moving with speed 4 m s^{-1} collides obliquely with a smooth sphere B of equal radius and of mass 2 kg. Just before the impact B is stationary and the velocity of A makes an angle of 30° with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{3}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

3 A smooth sphere A, of mass 3 kg and moving with speed 5 m s^{-1} collides obliquely with a smooth sphere B of equal radius and of mass 4 kg. Just before the impact B is stationary and the velocity of A makes an angle of 45° with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

A E/P **4** A small smooth sphere A of mass m and a small smooth sphere B of the same radius but mass $2m$ collide. At the instant of impact, B is stationary and the velocity of A makes an angle θ with the line of centres. The direction of motion of A is turned through 90° by the impact.
The coefficient of restitution between the spheres is e. Show that

$$\tan^2\theta = \frac{2e-1}{3}$$ **(8 marks)**

E **5** Two smooth spheres A and B are identical and are moving with equal speeds on a smooth horizontal surface. In the instant before impact, A is moving in a direction perpendicular to the line of centres of the spheres, and B is moving along the line of centres, as shown in the diagram. The coefficient of restitution between the spheres is $\frac{2}{3}$. Find the speeds and directions of motion of the spheres after the collision. **(8 marks)**

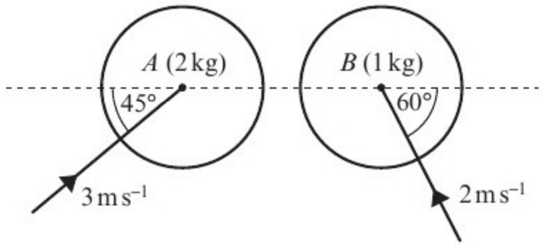


P **6** A smooth sphere A collides obliquely with an identical smooth sphere B. Just before the impact B is stationary and the velocity of A makes an angle of α with the line of centres of the two spheres. The coefficient of restitution between the spheres is $e\,(e \neq 1)$. Immediately after the collision the velocity of A makes an acute angle of β with the line of centres.

a Show that $\tan\beta = \dfrac{2\tan\alpha}{1-e}$

b Hence show that in the collision the direction of motion of A turns through an angle equal to $\arctan\left(\dfrac{(1+e)\tan\alpha}{2\tan^2\alpha + 1 - e}\right)$

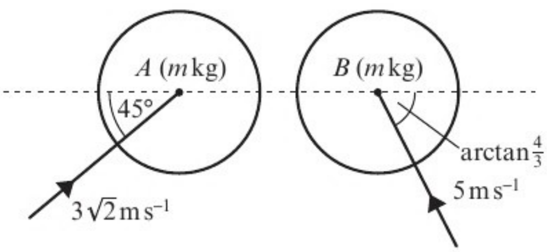
E **7** A small smooth sphere A of mass 2 kg collides with a small smooth sphere B of mass 1 kg. Just before the impact A is moving with a speed of 3 m s^{-1} in a direction at 45° to the line of centres and B is moving with speed 2 m s^{-1} at 60° to the line of centres, as shown in the diagram. The coefficient of restitution between the spheres is $\frac{\sqrt{2}}{3}$. Find:



a the kinetic energy lost in the impact **(8 marks)**

b the magnitude of the impulse exerted by A on B. **(2 marks)**

E **8** A small smooth sphere A collides with an identical small smooth sphere B. Just before the impact A is moving with a speed of $3\sqrt{2}$ m s^{-1} in a direction at 45° to the line of centres and B is moving with speed 5 m s^{-1} at an angle α to the line of centres, where $\tan\alpha = \frac{4}{3}$, as shown in the diagram.



A The coefficient of restitution between the spheres is $\frac{2}{3}$. Find:

a the speeds of both spheres immediately after the impact **(6 marks)**

b the fraction of the kinetic energy lost in the impact. **(3 marks)**

9 A smooth sphere A of mass 2 kg is moving on a smooth horizontal surface with velocity $(4\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$. Another smooth sphere B of mass 4 kg and the same radius as A is moving on the same surface with velocity $(2\mathbf{i} + \mathbf{j})\,\text{m s}^{-1}$. The spheres collide when their line of centres is parallel to $\mathbf{j}$. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the velocities of both spheres after the impact.

10 A smooth sphere A of mass 3 kg is moving on a smooth horizontal surface with velocity $(\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$. Another smooth sphere B of mass 1 kg and the same radius as A is moving on the same surface with velocity $(-4\mathbf{i} + 2\mathbf{j})\,\text{m s}^{-1}$. The spheres collide when their line of centres is parallel to $\mathbf{i}$. The coefficient of restitution between the spheres is $\frac{3}{4}$. Find the speeds of both spheres after the impact.

11 A smooth sphere A of mass 1 kg is moving on a smooth horizontal surface with velocity $(2\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$. Another smooth sphere B of mass 2 kg and the same radius as A is moving on the same surface with velocity $(-\mathbf{i} + \mathbf{j})\,\text{m s}^{-1}$. The spheres collide when their line of centres is parallel to $\mathbf{i}$. The coefficient of restitution between the spheres is $\frac{3}{5}$. Find the kinetic energy lost in the impact.

E **12** Two small smooth spheres A and B have equal radii. The mass of A is m kg and the mass of B is $2m$ kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i} + 5\mathbf{j})\,\text{m s}^{-1}$ and the velocity of B is $(3\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(3\mathbf{i} + 2\mathbf{j})\,\text{m s}^{-1}$. Find:

a the velocity of B immediately after the collision **(4 marks)**

b a unit vector parallel to the line of centres of the spheres at the instant of the collision. **(2 marks)**

E **13** Two small smooth spheres A and B have equal radii. The mass of A is $3m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(3\mathbf{i} - 5\mathbf{j})\,\text{m s}^{-1}$ and the velocity of B is $(4\mathbf{i} + \mathbf{j})\,\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(4\mathbf{i} - 4\mathbf{j})\,\text{m s}^{-1}$. Find:

a the speed of B immediately after the collision **(4 marks)**

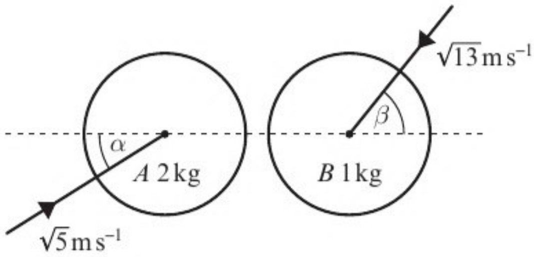
b the kinetic energy lost in the collision. **(3 marks)**

E **14** Two small smooth spheres A and B have equal radii. The mass of A is $2m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i} + 5\mathbf{j})\,\text{m s}^{-1}$ and the velocity of B is $(2\mathbf{i} - 2\mathbf{j})\,\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(3\mathbf{i} + 4\mathbf{j})\,\text{m s}^{-1}$. Find:

a the velocity of B immediately after the collision **(4 marks)**

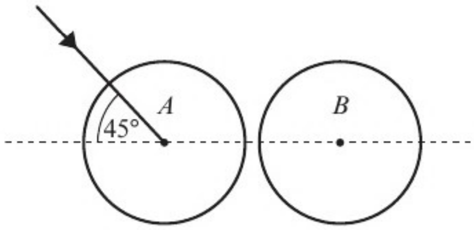
b the coefficient of restitution between the two spheres. **(3 marks)**

A E **15** Two smooth uniform spheres A and B of equal radius have masses 2 kg and 1 kg respectively. They are moving on a smooth horizontal plane when they collide. Immediately before the collision the speed of A is $\sqrt{5}$ m s^{-1} and the speed of B is $\sqrt{13}$ m s^{-1}. When they collide the line joining their centres makes an angle α with the direction of motion of A and an angle β with the direction of motion of B, where $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{3}{2}$, as shown in the diagram above. The coefficient of restitution between A and B is $\frac{1}{2}$.



Find the speed of each sphere after the collision. **(6 marks)**

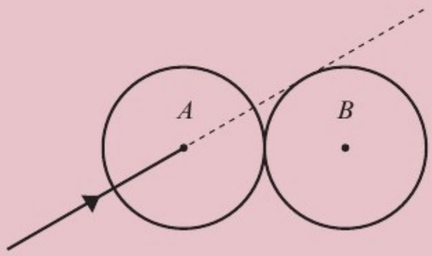
E/P **16** A smooth uniform sphere A, moving on a smooth horizontal table, collides with an identical sphere B which is at rest on the table. When the spheres collide the line joining their centres makes an angle of 45° with the direction of motion of A, as shown in the diagram. The coefficient of restitution between the spheres is e. The direction of motion of A is deflected through an angle θ by the collision.



Show that $\tan\theta = \dfrac{1+e}{3-e}$ **(10 marks)**

Challenge

A smooth uniform sphere B is at rest on a smooth horizontal plane, when it is struck by an identical sphere A moving on the plane. Immediately before the impact, the line of motion of the centre of A is tangential to the sphere B, as shown in the diagram above. The coefficient of restitution between the spheres is $\frac{1}{2}$. The direction of motion of A is turned through an angle θ by the impact.



Show that $\tan\theta = \dfrac{3\sqrt{3}}{7}$

Mixed exercise 5

E/P **1** A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of 45° with the wall. Immediately after the collision the speed of S is $\frac{4}{5}u$.

Find the coefficient of restitution between S and the wall. **(4 marks)**

E **2** A small smooth ball of mass $\frac{1}{2}$ kg is falling vertically. The ball strikes a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{5}{12}$. Immediately before striking the plane the ball has speed 5.2 m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{1}{4}$

Find:

a the speed, to 3 significant figures, of the ball immediately after the impact **(2 marks)**

b the magnitude of the impulse received by the ball as it strikes the plane. **(2 marks)**

A E/P **3** A small smooth ball of mass 500 g is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the line $x + y = 3$. The velocity of the ball just before the impact is $(-4\mathbf{i} - 2\mathbf{j})\,\text{m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{2}$. Find:

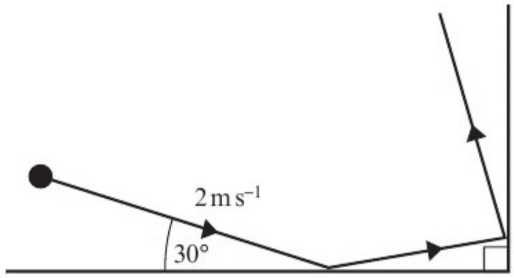
a the velocity of the ball immediately after the impact **(2 marks)**

b the kinetic energy lost as a result of the impact **(3 marks)**

c the angle of deflection of the ball. **(3 marks)**

E/P **4** A small smooth sphere is moving with velocity $(6\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(2\mathbf{i} - 2\mathbf{j})\,\text{m s}^{-1}$. Find the coefficient of restitution between the sphere and the wall. **(8 marks)**

E/P **5** Two walls stand on a floor and intersect at right angles. The walls and floor are modelled as smooth. A smooth sphere is moving across the surface and bounces off each wall in turn. At the moment just before the first impact the sphere has speed $2\,\text{m s}^{-1}$ at an angle of 30° to the wall. The coefficient of restitution between the sphere and the walls is 0.5. Find the speed and direction of motion of the sphere after:



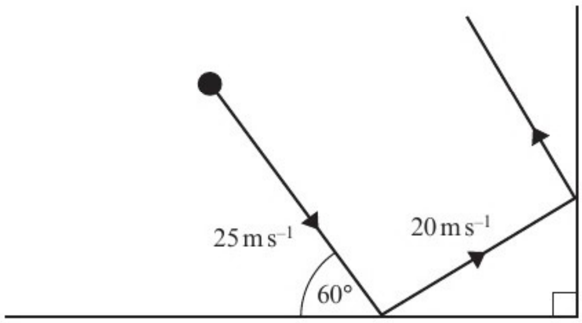
a the first collision **(4 marks)**

b the second collision. **(4 marks)**

In reality the floor and the walls may not be smooth.

c What effect will the model have had on the speed and direction of motion that you calculated in part **b**? **(2 marks)**

E/P **6** Two smooth vertical walls stand on a smooth horizontal surface and intersect at right angles. A smooth ball is moving in the xy-plane such that it collides with the first wall at a speed of $25\,\text{m s}^{-1}$ at an angle of 60° to the wall. The coefficient of restitution between the ball and both walls is e. Given that after the first collision the ball is moving with speed $20\,\text{m s}^{-1}$,

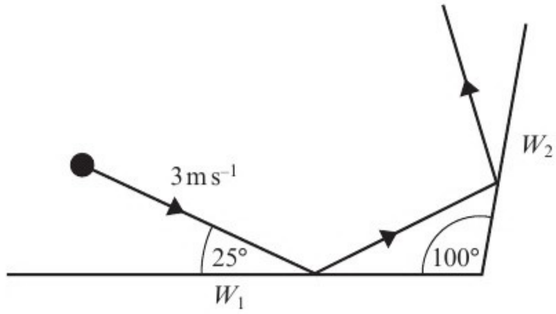


a find the value of e. **(4 marks)**

The ball then moves on to collide with the second wall.

b Calculate the speed of the ball after the second collision. **(4 marks)**

A E/P **7** Two vertical walls W_1 and W_2 stand on a smooth horizontal surface and intersect at an angle of 100°. A smooth sphere of mass 0.5 kg is projected across the surface with speed $3\,\text{m}\,\text{s}^{-1}$ at an angle of 25° to W_1 and towards the intersection of the walls. The coefficients of restitution between the sphere and walls W_1 and W_2 are 0.2 and 0.4 respectively.



a Find the kinetic energy lost in the first collision. **(5 marks)**

The sphere then moves on to collide with W_2.

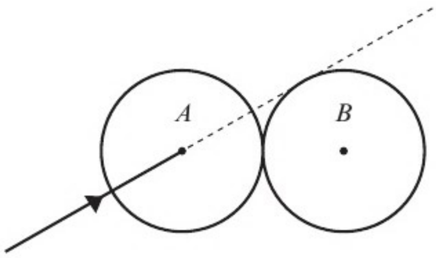
b Find the speed and direction of motion of the sphere after the second collision. **(4 marks)**

E/P **8** Two small smooth spheres A and B have equal radii. The mass of A is $4m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i} + 3\mathbf{j})\,\text{m}\,\text{s}^{-1}$ and the velocity of B is $(3\mathbf{i} - \mathbf{j})\,\text{m}\,\text{s}^{-1}$. Immediately after the collision the velocity of A is $(3\mathbf{i} + 2\mathbf{j})\,\text{m}\,\text{s}^{-1}$. Find:

a the velocity of B immediately after the collision **(2 marks)**

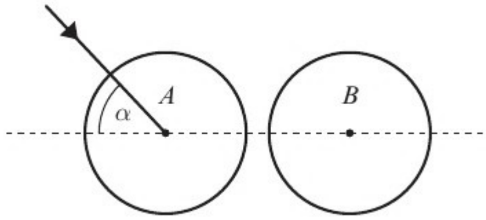
b a unit vector parallel to the line of centres of the spheres at the instant of the collision. **(3 marks)**

E **9** A smooth uniform sphere B is at rest on a smooth horizontal plane, when it is struck by an identical sphere A moving on the plane. Immediately before the impact, the line of motion of the centre of A is tangential to the sphere B, as shown in the diagram above. The coefficient of restitution between the spheres is $\frac{2}{3}$. The direction of motion of A is turned through an angle θ by the impact.



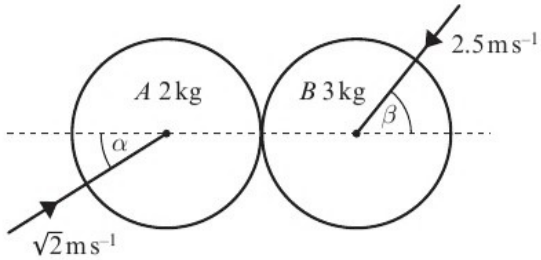
Show that $\tan\theta = \frac{5\sqrt{3}}{9}$ **(8 marks)**

E/P **10** A smooth uniform sphere A, moving on a smooth horizontal table, collides with a second identical sphere B which is at rest on the table. When the spheres collide the line joining their centres makes an angle of α with the direction of motion of A, as shown in the diagram. The direction of motion of A is deflected through an angle θ by the collision.



Given that $\tan\alpha = \frac{3}{4}$ and that the coefficient of restitution between the spheres is e, show that $\tan\theta = \frac{6 + 6e}{17 - 8e}$ **(8 marks)**

A **E/P** **11** Two smooth uniform spheres A and B of equal radius have masses 2 kg and 3 kg respectively. They are moving on a smooth horizontal plane when they collide. Immediately before the collision the speed of A is $\sqrt{2}\,\text{m s}^{-1}$ and the speed of B is $2.5\,\text{m s}^{-1}$. When they collide the line joining their centres makes an angle α with the direction of motion of A and an angle β with the direction of motion of B, where $\tan\alpha = 1$ and $\tan\beta = \frac{3}{4}$, as shown in the diagram.

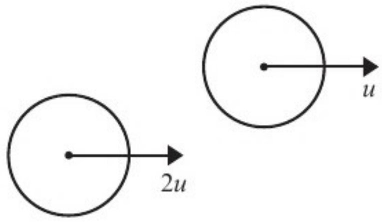


The coefficient of restitution between A and B is $\frac{2}{3}$

Find the speed of each sphere after the collision. **(8 marks)**

E/P **12** A red ball is stationary on a rectangular billiard table $OABC$. It is then struck by a white ball of equal mass and equal radius moving with velocity $u(-2\mathbf{i} + 8\mathbf{j})\,\text{m s}^{-1}$ where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors parallel to OA and OC respectively. After the impact the velocity of the red ball is parallel to the vector $(-\mathbf{i} + \mathbf{j})$ and the velocity of the white ball is parallel to the vector $(2\mathbf{i} + 4\mathbf{j})$. Prove that the coefficient of restitution between the two balls is $\frac{3}{5}$ **(8 marks)**

E/P **13** Two uniform spheres, each of mass m and radius a, collide when moving on a horizontal plane. Before the impact the spheres are moving with speeds $2u$ and u, as shown in the diagram.



The centres of the spheres are moving on parallel paths a perpendicular distance $\frac{6a}{5}$ apart.

The coefficient of restitution between the spheres is $\frac{3}{4}$. Find the speeds of the spheres just after the impact, and show that the angle between their paths is then equal to $\arctan\frac{14}{23}$ **(10 marks)**

Challenge

Two smooth vertical walls stand on a smooth horizontal surface and intersect at an angle of 45°. A smooth sphere of mass 0.5 kg is projected across the surface with speed $2\,\text{m s}^{-1}$ at an angle of θ to one of the walls and towards the intersection of the walls, where $\tan\theta = \frac{10}{7}$

The coefficient of restitution between the sphere and the walls is $\frac{1}{2}$

a Show that there will be exactly three collisions between the sphere and a wall.

b Find the proportion of the initial kinetic energy that is lost in the three collisions.

Summary of key points

A

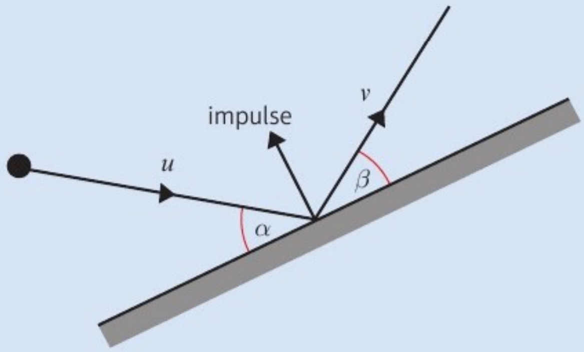
1 In an **oblique** impact between a smooth sphere and a smooth fixed surface:

- The impulse on the sphere acts perpendicular to the surface, through the centre of the sphere.
- The component of the velocity of the sphere parallel to the surface is unchanged.

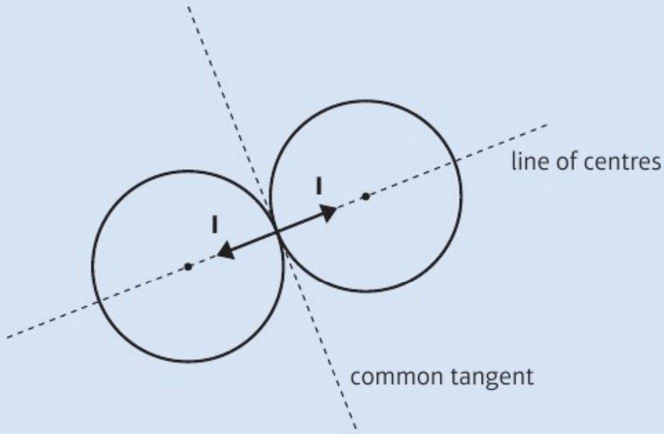
  $v \cos \beta = u \cos \alpha$

- You can use **Newton's law of restitution** to find the component of the velocity of the sphere perpendicular to the surface.

  $v \sin \beta = eu \sin \alpha$



2 In an impact between two spheres:



- The reaction between the two spheres acts along the **line of centres**, so the impulse affecting each sphere also acts along the line of centres.
- The components of the velocities of the spheres perpendicular to the line of centres are unchanged in the impact.
- Newton's law of restitution applies to the components of the velocities of the spheres parallel to the line of centres.
- The principle of conservation of momentum applies parallel to the line of centres.

Review exercise 2

(E/P) **1** A smooth sphere S of mass m is moving on a smooth horizontal plane with speed u. It collides directly with another smooth sphere T, of mass $3m$, whose radius is the same as S. The sphere T is moving in the same direction as S with speed $\frac{1}{6}u$. The sphere S is brought to rest by the impact. Find the coefficient of restitution between the spheres. **(7)**

← Section 4.1

(E/P) **2** A smooth sphere S of mass m is moving with speed u on a smooth horizontal plane. The sphere S collides with another smooth sphere T, of equal radius to S but of mass km, moving in the same straight line and in the same direction with speed λu, $0 < \lambda < \frac{1}{2}$. The coefficient of restitution between S and T is e.

Given that S is brought to rest by the impact,

a show that $e = \dfrac{1 + k\lambda}{k(1 - \lambda)}$ **(6)**

b deduce that $k > 1$. **(3)**

← Section 4.1

(E/P) **3** A smooth uniform sphere S of mass m is moving on a smooth horizontal plane with speed u. The sphere collides directly with another smooth uniform sphere T, of the same radius as S and of mass $2m$, which is at rest on the plane. The coefficient of restitution between the spheres is e.

a Show that the speed of T after the collision is $\frac{1}{3}u(1 + e)$.

Given that $e > \frac{1}{2}$,

b i find the speed of S after the collision **(4)**

ii determine whether the direction of motion of S is reversed by the collision. **(4)**

← Section 4.1

(E/P) **4** A particle P of mass $3m$ is moving with speed $2u$ in a straight line on a smooth horizontal table. The particle P collides with a particle Q of mass $2m$ moving with speed u in the opposite direction to P. The coefficient of restitution between P and Q is e.

a Show that the speed of Q after the collision is $\frac{1}{5}u(9e + 4)$. **(5)**

As a result of the collision, the direction of motion of P is reversed.

b Find the range of possible values of e. **(5)**

Given that the magnitude of the impulse of P on Q is $\frac{32}{5}mu$,

c find the value of e. **(4)**

← Section 4.1

(E) **5** A ball falls from rest and hits a smooth horizontal plane 2 seconds later. Given that the coefficient of restitution between the ball and the plane is $\frac{6}{7}$, find the maximum height reached by the ball on its first bounce. **(4)**

← Section 4.2

E/P **6** A small sphere S is projected from a point P along a smooth horizontal plane towards a fixed vertical wall, which is perpendicular to the direction of motion of the sphere. S strikes the wall 2 seconds after it is projected, and passes through P again 3 seconds after that.

a Find the value of the coefficient of restitution between the sphere and the wall. **(3)**

b Without further calculation, state with justification how your answer to part **a** would change if the plane was rough. **(2)**

← Section 4.2

E **7** A ball B falls from rest from a cliff of height 50 m onto horizontal ground. After rebounding the ball reaches a maximum height of 35 m.

a Find the value of the coefficient of restitution between the ball and the ground. **(3)**

b Show that the exact time taken from the instant when the ball was dropped until the instant when it hits the ground for a second time is $\frac{10}{7}(\sqrt{5} + \sqrt{14})$ s **(4)**

← Section 4.2

E/P **8** Two particles, A and B, of mass m and $3m$ respectively, lie at rest on a smooth horizontal table. The coefficient of restitution between the particles is 0.25. A and B are moving towards each other at speeds of $7u$ and u respectively and they collide directly. Find:

a the speeds of A and B after the collision **(7)**

b the loss in kinetic energy due to the collision. **(2)**

← Sections 4.1, 4.3

E/P **9** Two uniform smooth spheres A and B are of equal size and have masses $3m$ and $2m$ respectively. They are both moving in the same straight line with speed u, but in opposite directions, when they are in direct collision with each other. Given that A is brought to rest by the collision, find:

a the coefficient of restitution between the spheres **(6)**

b the kinetic energy lost in the impact. **(3)**

← Sections 4.1, 4.3

E/P **10** A smooth sphere A of mass m is moving with speed u on a smooth horizontal table when it collides directly with another smooth sphere B of mass $3m$, which is at rest on the table. The coefficient of restitution between A and B is e. The spheres have equal radius and are modelled as particles.

a Show that the speed of B immediately after the collision is $\frac{1}{4}(1 + e)u$ **(5)**

b Find the speed of A immediately after the collision. **(2)**

Immediately after the collision the total kinetic energy of the spheres is $\frac{1}{6}mu^2$

c Find the value of e. **(6)**

d Hence show that A is at rest after the collision. **(1)**

← Sections 4.1, 4.3

E **11** A train engine of mass 8 tonnes is moving at $4\,\text{m}\,\text{s}^{-1}$ when it hits a carriage of mass 12 tonnes which is at rest. After the impact, the engine and the carriage move off together with speed $1.5\,\text{m}\,\text{s}^{-1}$ in the direction in which the engine was originally moving. Calculate the total loss of kinetic energy due to the impact. **(4)**

← Section 4.3

E/P **12** Two particles, P and Q, of masses 0.05 kg and 0.25 kg respectively, are connected by a light inextensible string. P is projected with speed $2\,\text{m s}^{-1}$ directly away from Q. When the string becomes taut, particle Q is jerked into motion and P and Q then move with a common speed in the direction in which P was originally moving. Find the loss of total kinetic energy due to the jerk. **(10)**

← Section 4.3

E/P **13** Two small spheres A and B have masses $3m$ and $2m$ respectively. They are moving towards each other in opposite directions on a smooth horizontal plane, both with speed $2u$, when they collide directly. As a result of the collision, the direction of motion of B is reversed and its speed is unchanged.

a Find the coefficient of restitution between the spheres. **(7)**

Subsequently, B collides directly with another small sphere C of mass $5m$ which is at rest. The coefficient of restitution between B and C is $\frac{3}{5}$

b Show that, after B collides with C, there will be no further collisions between the spheres. **(7)**

← Sections 4.1, 4.4

E/P **14** A smooth sphere P of mass $2m$ is moving in a straight line with speed u on a smooth horizontal table. Another smooth sphere Q of mass m is at rest on the table. P collides directly with Q. The coefficient of restitution between P and Q is $\frac{1}{3}$. The spheres are modelled as particles.

a Show that, immediately after the collision, the speeds of P and Q are $\frac{5}{9}u$ and $\frac{8}{9}u$ respectively. **(7)**

After the collision, Q strikes a fixed vertical wall which is perpendicular to the direction of motion of P and Q. The coefficient of restitution between Q and the wall is e. When P and Q collide again, P is brought to rest.

b Find the value of e. **(7)**

c Explain why there must be a third collision between P and Q. **(1)**

← Sections 4.1, 4.2, 4.4

E **15** Two small smooth spheres, P and Q, of equal radius, have masses $2m$ and $3m$ respectively. The sphere P is moving with speed $5u$ on a smooth horizontal table when it collides directly with Q, which is at rest on the table. The coefficient of restitution between P and Q is e.

a Show that the speed of Q immediately after the collision is $2(1 + e)u$ **(5)**

After the collision, Q hits a smooth vertical wall which is at the edge of the table and perpendicular to the direction of motion of Q. The coefficient of restitution between Q and the wall is f, $0 < f \leqslant 1$.

b Show that, when $e = 0.4$, there is a second collision between P and Q. **(3)**

Given that $e = 0.8$ and there is a second collision between P and Q,

c find the range of possible values of f. **(3)**

← Sections 4.1, 4.2, 4.4

E/P **16** A particle A of mass $2m$, moving with speed $2u$ in a straight line on a smooth horizontal table, collides with a particle B of mass $3m$, moving with speed u in the same direction as A. The coefficient of restitution between A and B is e.

a Show that the speed of B after the collision is

$\frac{1}{5}u(7 + 2e)$ **(5)**

b Find the speed of A after the collision, in terms of u and e. **(2)**

The speed of A after the collision is $\frac{11}{10}u$.

c Show that $e = \frac{1}{2}$ **(2)**

At the instant of collision, A and B are at a distance d from a vertical barrier fixed to the surface at right-angles to their direction of motion. Given that B hits the barrier, and that the coefficient of restitution between B and the barrier is $\frac{11}{16}$,

d find the distance of A from the barrier at the instant that B hits the barrier **(4)**

e show that, after B rebounds from the barrier, it collides with A again at a distance $\frac{5}{32}d$ from the barrier. **(4)**

← Sections 4.1, 4.2, 4.4

E/P **17** A ball is dropped from a height of 2 m onto a smooth horizontal plane. The coefficient of restitution between the ball and the plane is 0.8.

a Modelling the ball as a particle, find the total distance travelled by the ball before it comes to rest. **(10)**

b Criticise this model with respect to the number of times the ball bounces. **(1)**

← Section 4.4

E/P **18** Three small smooth spheres A, B and C of masses m, $2m$ and $3m$ respectively lie in order in a straight line on a smooth horizontal surface. The coefficient of restitution between A and B is 0.7 and the coefficient of restitution between B and C is 0.4. Sphere A is projected towards sphere B with speed $4\,\text{m s}^{-1}$. Show that exactly two collisions will occur. **(12)**

← Section 4.4

A E **19** A smooth uniform sphere S of mass m is moving on a smooth horizontal plane when it collides with a fixed smooth vertical wall. Immediately before the collision, the speed of S is U and its direction of motion makes an angle α with the wall. The coefficient of restitution between S and the wall is e.

Find the kinetic energy of S immediately after the collision. **(6)**

← Section 5.1

A E **20** A small ball is moving on a horizontal plane when it strikes a smooth vertical wall. The coefficient of restitution between the ball and the wall is e. Immediately before the impact the direction of motion of the ball makes an angle of 60° with the wall. Immediately after the impact the direction of motion of the ball makes an angle of 30° with the wall.

a Find the fraction of the kinetic energy of the ball which is lost in the impact. **(6)**

b Find the value of e. **(4)**

← Section 5.1

E/P **21** A smooth sphere slides across a smooth horizontal plane, striking a vertical wall with a speed of $\sqrt{3}v$ and at an angle of θ to the wall. After striking the wall the sphere moves at right-angles to its original direction of motion with speed v.

a Find the value of θ. **(3)**

b Show that the coefficient of restitution between the sphere and the wall is $\frac{1}{3}$. **(4)**

← Section 5.1

E/P **22** A small smooth ball is dropped from a height of 20 m above a plane that is inclined at an angle of θ to the horizontal, where $\tan\theta = \frac{1}{\sqrt{3}}$. Immediately after the impact the ball has speed $15\,\text{m s}^{-1}$. Find the coefficient of restitution between the ball and the plane. **(6)**

← Section 5.1

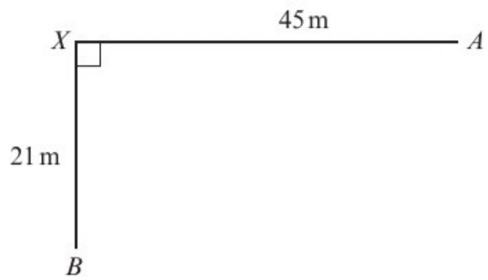
E **23** A smooth sphere of mass 0.5 kg is moving with velocity $(3\mathbf{i} + 2\mathbf{j})\,\text{m s}^{-1}$ when it hits a smooth wall. It rebounds with velocity $(2\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$. Find:

a the magnitude and direction of the impulse received by the sphere **(3)**

b the kinetic energy lost in the collision. **(3)**

← Section 5.1

A E/P **24**



The figure represents the scene of a road accident. A car of mass 600 kg collided at the point X with a stationary van of mass 800 kg. After the collision the van came to rest at the point A having travelled a horizontal distance of 45 m, and the car came to rest at the point B having travelled a horizontal distance of 21 m. The angle AXB is 90°.

The accident investigators are trying to establish the speed of the car before the collision and they model both vehicles as small spheres.

a Find the coefficient of restitution between the car and the van. **(5)**

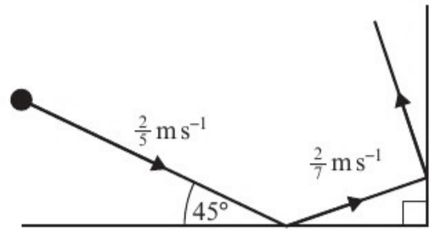
The investigators assume that after the collision, and until the vehicles came to rest, the van was subject to a constant horizontal force of 500 N acting along AX and the car to a constant horizontal force of 300 N along BX.

b Find the speed of the car immediately before the collision. **(9)**

← Section 5.3

E/P **25** Two smooth vertical walls stand on a smooth horizontal surface and intersect at right angles. A smooth sphere of mass 0.8 kg is moving across the surface such that it collides with the first wall at a speed of $\frac{2}{5}\,\text{m s}^{-1}$ at an angle of 45° to the wall. The coefficient of restitution between the sphere and both walls is e. After the first collision, the sphere is moving with speed $\frac{2}{7}\,\text{m s}^{-1}$.

A



Find:

a the direction in which the sphere is moving after the first impact **(2)**

b the value of e. **(2)**

The sphere then moves on to collide with the second wall.

c Calculate the kinetic energy of the sphere after the second collision. **(6)**

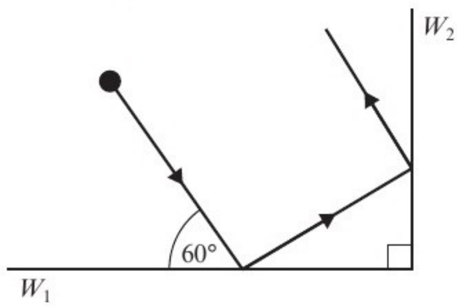
← Section 5.2

E/P **26** Two vertical walls stand on a smooth horizontal surface and intersect at an angle of 80°. A smooth sphere of mass 0.3 kg is projected across the surface with speed $2\,\text{m s}^{-1}$ at an angle of 30° to one of the walls and towards the intersection of the walls. The sphere then collides with both walls. The coefficient of restitution between the sphere and the walls is 0.6.

Work out the total kinetic energy lost during the two collisions. **(8)**

← Section 5.2

E/P **27** Two smooth vertical walls, W_1 and W_2, stand on a smooth horizontal surface and intersect at right angles. A small smooth sphere is moving with speed 4 m s^{-1} when it hits W_1 at an angle of 60°. It rebounds from the wall with speed 3 m s^{-1} and goes on to hit W_2.



a Find the coefficient of restitution between the sphere and W_1. **(4)**

A

Assuming that the coefficient of restitution between the sphere and W_2 is 0.35,

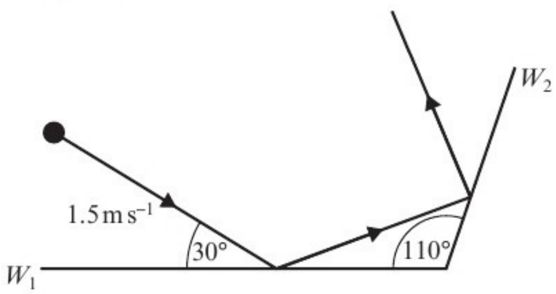
b work out the speed of the sphere and direction in which it is moving after it collides with W_2. **(6)**

← Section 5.2

E/P **28** Two smooth vertical walls stand on a smooth horizontal surface and intersect at right angles. A small smooth sphere of mass m is moving with velocity $(4\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$ when it hits one of the walls. It rebounds from the wall with velocity $(\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$ and goes on to hit the second wall. Given that the coefficient of restitution between the sphere and each wall is the same, find the total kinetic energy lost by the sphere in both collisions. **(8)**

← Section 5.2

E/P **29** Two smooth vertical walls, W_1 and W_2, stand on a smooth horizontal surface and intersect at an angle of 110°. A smooth sphere of mass 1.6 kg is projected across the surface with speed $1.5\,\text{m s}^{-1}$ at an angle of 30° to wall W_1 and towards the intersection of the walls. The coefficient of restitution between the sphere and wall W_1 is 0.8.



a Work out the speed and direction of motion of the sphere after the first collision. **(6)**

The sphere then moves on to collide with W_2. Given that after the second collision, the sphere has kinetic energy 1.35 J,

b work out the coefficient of restitution between the sphere and wall W_2. **(8)**

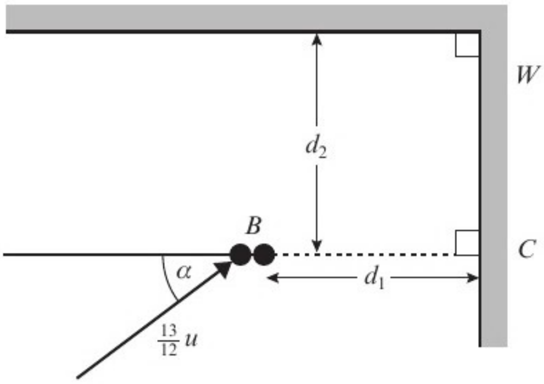
← Section 5.2

A E/P **30** Two vertical walls W_1 and W_2 stand on a smooth horizontal surface and intersect at an angle of 100°. A smooth sphere of mass 1.7 kg is projected across the surface with speed $8\,\text{m s}^{-1}$ at an angle of 25° to wall W_1 and towards the intersection of the walls. The coefficient of restitution between the sphere and walls W_1 and W_2 is 0.6 and 0.7 respectively.

Calculate the total kinetic energy lost by the sphere. **(10)**

← Section 5.2

E/P **31**



A small ball Q of mass $2m$ is at rest at the point B on a smooth horizontal plane. A second small ball P of mass m is moving on the plane with speed $\frac{13}{12}u$ and collides with Q. Both the balls are smooth, uniform and of equal radius. The point C is on a smooth vertical wall W which is at a distance d_1 from B, and BC is perpendicular to W. A second smooth vertical wall is perpendicular to W and at a distance d_2 from B. Immediately before the collision occurs, the direction of motion of P makes an angle α with BC, as shown in the figure, where $\tan\alpha = \frac{5}{12}$. The line of centres of P and Q is parallel to BC. After the collision Q moves towards C with speed $\frac{3}{5}u$.

a Show that, after the collision, the velocity components of P parallel and perpendicular to CB are $\frac{1}{5}u$ and $\frac{5}{12}u$ respectively. **(4)**

A

b Find the coefficient of restitution between P and Q. **(2)**

c Show that when Q reaches C, P is at a distance $\frac{4}{3}d_1$ from W. **(3)**

For each collision between a ball and a wall the coefficient of restitution is $\frac{1}{2}$

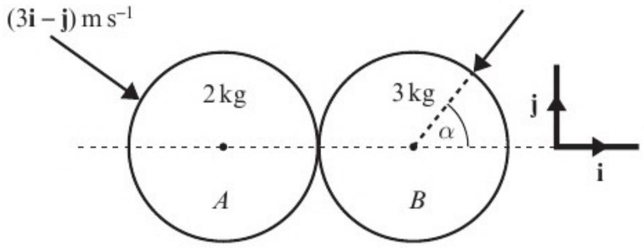
Given that the balls collide with each other again,

d show that the time between the two collisions of the balls is $\frac{15d_1}{u}$ **(4)**

e find the ratio $d_1 : d_2$ **(5)**

← Section 5.3

E/P **32**

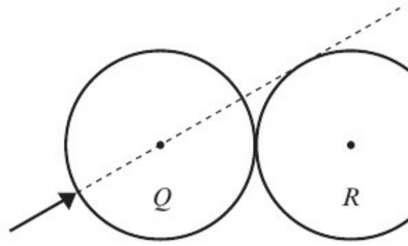


Two smooth uniform spheres A and B, of equal radius, are moving on a smooth horizontal plane. Sphere A has mass 2 kg and sphere B has mass 3 kg. The spheres collide and at the instant of collision the line joining their centres is parallel to **i**. Before the collision A has velocity $(3\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$ and after the collision it has velocity $(-2\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$. Before the collision the velocity of B makes an angle α with the line of centres, as shown in the figure, where $\tan\alpha = 2$. The coefficient of restitution between the spheres is $\frac{1}{2}$.

Find, in terms of **i** and **j**, the velocity of B before the collision. **(9)**

← Section 5.3

A E/P **33**

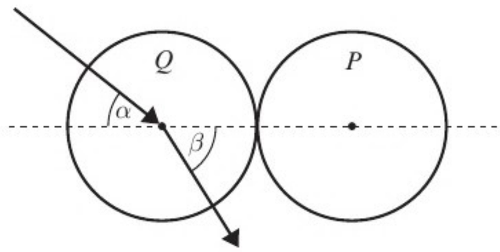


A smooth uniform sphere R is at rest on a smooth horizontal plane, when it is struck by an identical sphere Q moving on the plane. Immediately before the impact, the line of motion of the centre of Q is tangential to the sphere R, as shown in the figure. The direction of motion of Q is turned through 30° by the impact.

Find the coefficient of restitution between the spheres. **(11)**

← Section 5.3

E/P **34**

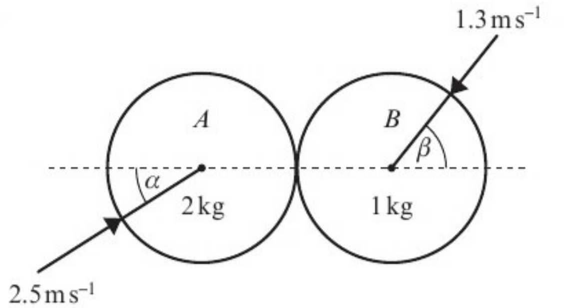


A smooth sphere P lies at rest on a smooth horizontal plane. A second identical sphere Q, moving on the plane, collides with P. Immediately before the collision the direction of motion of Q makes an angle α with the line joining the centres of the spheres. Immediately after the collision, the direction of motion of Q makes an angle β with the line joining the centres of the spheres, as shown in the figure. The coefficient of restitution between the spheres is e.

Show that $(1 - e)\tan\beta = 2\tan\alpha$ **(11)**

← Section 5.3

A E/P **35**



Two smooth uniform spheres A and B of equal radius have masses 2 kg and 1 kg respectively. They are moving on a smooth horizontal plane when they collide. Immediately before the collision the speed of A is $2.5\,\text{m s}^{-1}$ and the speed of B is $1.3\,\text{m s}^{-1}$. When they collide, the line joining their centres makes an angle α with the direction of motion of A and an angle β with the direction of motion of B, where $\tan\alpha = \frac{4}{3}$ and $\tan\beta = \frac{12}{5}$, as shown in the figure.

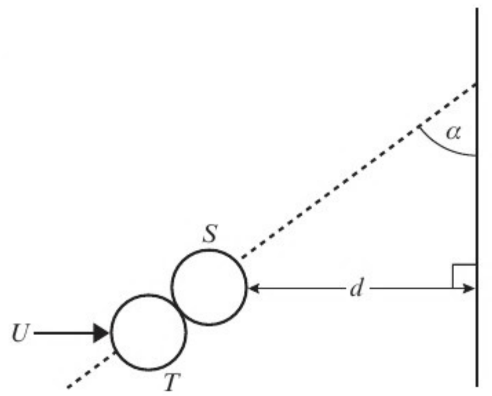
a Find the components of the velocities of A and B perpendicular and parallel to the line of centres immediately before the collision. **(4)**

The coefficient of restitution between A and B is $\frac{1}{2}$

b Find, to one decimal place, the speed of each sphere after the collision. **(9)**

← Section 5.3

E/P **36**



A small smooth uniform sphere S is at rest on a smooth horizontal floor at a distance d from a straight vertical wall. An identical sphere T is projected along the floor with speed U towards S and in a direction which is perpendicular to the wall. At the instant when T strikes S the line joining their centres makes an angle α with the wall, as shown in the figure.

Each sphere is modelled as having negligible diameter in comparison with d. The coefficient of restitution between the spheres is e.

a Show that the components of the velocity of T after the impact, parallel and perpendicular to the line of centres, are $\frac{1}{2}U(1-e)\sin\alpha$ and $U\cos\alpha$ respectively. **(7)**

b Show that the components of the velocity of T after the impact, parallel and perpendicular to the wall are $\frac{1}{2}U(1+e)\cos\alpha\sin\alpha$ and $\frac{1}{2}U[2-(1+e)\sin^2\alpha]$ respectively. **(6)**

The spheres S and T strike the wall at the points A and B respectively.

Given that $e = \frac{2}{3}$ and $\tan\alpha = \frac{3}{4}$,

c find, in terms of d, the distance AB. **(5)**

← Section 5.3

E/P **37** A smooth uniform sphere A has mass $2m$ kg and another smooth uniform sphere B, with equal radius to A, has mass m kg. The spheres are moving on a smooth horizontal plane when they collide. At the instant of collision the line joining the centres of the spheres is parallel to **j**. Immediately after the collision, the velocity of A is $(3\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$ and the velocity of B is $(2\mathbf{i} + \mathbf{j})\,\text{m s}^{-1}$. The coefficient of restitution between the spheres is $\frac{1}{2}$.

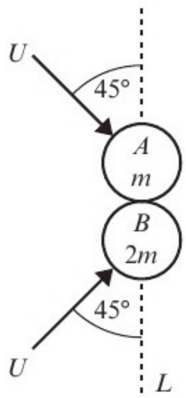
a Find the velocities of the two spheres immediately before the collision. **(7)**

b Find the magnitude of the impulse in the collision. **(2)**

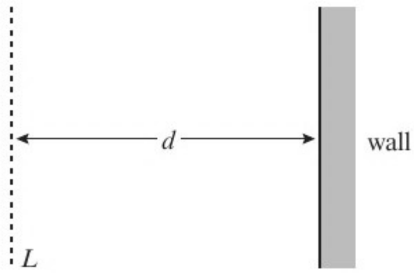
c Find, to the nearest degree, the angle through which the direction of motion of A is deflected by the collision. **(4)**

← Section 5.3

A E/P **38** Two small spheres A and B, of equal size and of masses m and $2m$ respectively, are moving initially with the same speed U on a smooth horizontal floor. The spheres collide when their centres are on a line L. Before the collision the spheres are moving towards each other, with their directions of motion perpendicular to each other and each inclined at an angle 45° to the line L, as shown in the figure below. The coefficient of restitution between the spheres is $\frac{1}{2}$



a Find the magnitude of the impulse which acts on A in the collision. **(9)**



The line L is parallel to and a distance d from a smooth vertical wall, as shown in the second figure.

b Find, in terms of d, the distance between the points at which the spheres first strike the wall. **(5)**

← Section 5.3

A E/P **39** Two smooth uniform spheres A and B have equal radii. Sphere A has mass m and sphere B has mass km. The spheres are at rest on a smooth horizontal table. Sphere A is then projected along the table with speed u and collides with B. Immediately before the collision, the direction of motion of A makes an angle of 60° with the line joining the centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{2}$

a Show that the speed of B immediately after the collision is $\frac{3u}{4(k+1)}$ **(6)**

Immediately after the collision the direction of motion of A makes an angle $\arctan(2\sqrt{3})$ with the direction of motion of B.

b Show that $k = \frac{1}{2}$ **(6)**

c Find the loss of kinetic energy due to the collision. **(4)**

Section 5.3

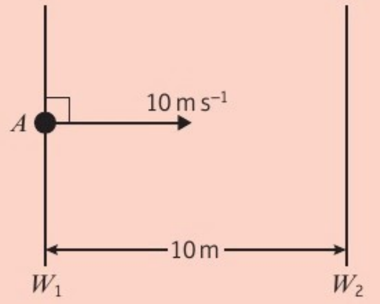
Challenge

1 Two balls A and B of equal mass, lie on a smooth horizontal plane between two smooth, parallel, vertical walls W_1 and W_2 that lie l m apart. Ball A is initially in contact with W_1 and ball B is initially in contact with W_2. The balls are projected towards each other at right-angles to the walls, such that A has speed $6u\,\text{m s}^{-1}$ and B has speed $5u\,\text{m s}^{-1}$. The balls collide directly, and the coefficient of restitution between the balls is e. Show that the total time taken, in seconds, before ball A comes into contact with W_1 again is given by

$$\frac{l(e+1)}{u(11e-1)}$$

← Section 4.1

2 A ball is projected at a speed of $10\,\text{m s}^{-1}$ from a point A on a smooth vertical wall W_1 across a smooth horizontal surface. It travels in a perpendicular direction to W_1 before striking a second parallel wall W_2 which lies a distance 10 m from W_1.



a Given that the coefficient of restitution between the sphere and each wall is 0.8 find:

i the time taken for the sphere to first return to A

ii the further time taken for the sphere to return to A a second time.

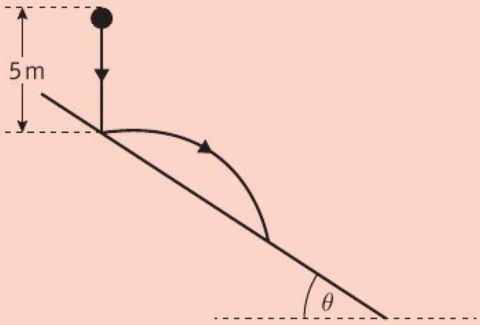
b Find the total time taken from the moment the ball is projected to the time when it returns to A for the 20th time, and its speed at that moment.

c Criticise the model with respect to your answer to part **b**.

← Section 4.2

A

3 A ball is dropped from a height of 5 m and bounces on a smooth plane inclined at an angle of θ to the horizontal, where $\sin\theta = \frac{3}{5}$.



Given that the coefficient of restitution between the ball and the plane is 0.5, and taking $g = 10\,\text{m s}^{-2}$,

a find the distance between the point at which the ball bounces for the first time and the point at which it bounces for the second time

b show that the ball loses 48% of its kinetic energy in the first bounce, but only 7.5% of its kinetic energy in the second bounce, and briefly justify the difference.

← Sections 5.1, 5.2

Exam-style practice

Further Mathematics
AS Level
Further Mechanics 1

Time: 50 minutes
You must have: Mathematical Formulae and Statistical Tables, Calculator

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1 A ball of mass 2.5 kg is dropped from a height of 12 m, and is travelling at a speed of 10 m s^{-1} immediately before it hits the ground. The falling ball is subject to air resistance, which is modelled as a constant force of magnitude R N.

a Find:

i the work done by air resistance

ii the magnitude of R. **(4)**

The model for air resistance is refined so that it is modelled as a variable force of magnitude $(10 + 0.2v^2)$ N, where v m s^{-1} is the speed of the ball.

b Find the maximum possible velocity of the ball according to the new model. **(2)**

2 A train engine of mass 5000 kg pulls a carriage of mass 10 000 kg with constant speed v m s^{-1} along a track that is angled at θ to the horizontal, where $\sin\theta = \frac{1}{50}$. The engine is working at a rate of 40 kW. The resistance to motion from non-gravitational forces has magnitude 2500 N. Find the value of v. **(5)**

3 Two particles, A and B, of equal radius, lie at rest on a smooth horizontal table. A has mass $2m$ and B has mass m. Particle A is projected towards B with speed 2 m s^{-1} and collides directly with B. The coefficient of restitution between the particles is 0.8.

a Find the velocities of A and B after this collision. **(7)**

b Given that 0.36 J of kinetic energy is lost during the collision, work out the value of m. **(4)**

4 A particle of mass m kg lies on a smooth horizontal surface between a pair of fixed parallel vertical walls W_1 and W_2 that are a distance of l m apart. Initially the particle is at rest and in contact with W_1. At time $t = 0$ the particle is projected from W_1 with speed u m s^{-1} in a direction perpendicular to the walls. The coefficient of restitution between the particle and each wall is e. The magnitude of the impulse on the particle due to the first impact with a wall is λ N s.

a Show that $\lambda = mu(1 + e)$. **(4)**

After bouncing off W_2, the particle returns to W_1 at time T s.

b Show that $T = \frac{l}{eu}(e + 1)$. **(4)**

5 Three balls, A, B and C, of equal radius and masses of m kg, $2m$ kg and $3m$ kg respectively, lie at rest in a straight line on a smooth horizontal table. Ball A is projected towards ball B with speed 3 m s^{-1}. The coefficient of restitution between each pair of balls is 0.9. Show that there are exactly two collisions. **(10)**

Exam-style practice

Further Mathematics A Level Further Mechanics 1

Time: 1 hour and 30 minutes
You must have: Mathematical Formulae and Statistical Tables, Calculator

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1 A particle P of mass 6 kg is travelling along rough horizontal ground when it collides with a particle Q of mass 4 kg. Immediately before the collision P is travelling with a speed of $2.5\,\text{m s}^{-1}$. After the collision the two particles coalesce.

a State the coefficient of restitution between the two particles. **(1)**

b Find the speed of the combined particles immediately after the collision. **(2)**

c Calculate the kinetic energy lost in the collision. **(3)**

d Given that combined particles come to rest after travelling 2 m, find the coefficient of friction between the particles and the ground. **(3)**

2 A car of mass 1400 kg is moving along a straight horizontal road. When the car is travelling at a speed of $v\,\text{m s}^{-1}$, its total non-gravitational resistances to motion are modelled as a variable force of magnitude $(120 + 2v^2)$ N.

The engine of the car is working at a constant rate of 20 kW.

a Find the acceleration of the car at the instant when $v = 16$. **(4)**

The car now travels downhill on a straight road at an angle of 6° to the horizontal. The driver wishes to maintain a constant speed of $20\,\text{m s}^{-1}$.

b Show that the driver will need to brake to maintain this speed. **(3)**

The driver places the car in neutral so that it freewheels with no driving force and no braking force.

c Find the maximum speed of the car. **(4)**

3 A ball of mass m kg is dropped from a vertical height of h m above a smooth plane that is inclined at an angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$.

The coefficient of restitution between the ball and the plane is e.

Given that the ball loses half of its kinetic energy on impact with the plane, find the value of e.

(9)

4 A football of mass 0.2 kg is travelling with velocity $(5\mathbf{i} - 2\mathbf{j})\,\text{m s}^{-1}$ when it receives an impulse of $\mathbf{P}$ N. Immediately afterwards its velocity is $(8\mathbf{i} + 4\mathbf{j})\,\text{m s}^{-1}$. Find:

a the magnitude of $\mathbf{P}$

b the angle between $\mathbf{P}$ and $\mathbf{i}$. **(5)**

5 An elastic string has natural length 1.2 m and modulus of elasticity 15 N. One end is fixed at a point P on a horizontal ceiling. A ball of mass 0.25 kg is attached to the free end and hangs in equilibrium at the point Q, vertically below P.

a Find the distance PQ. **(3)**

The string is then stretched to a length of 1.9 m.

b Calculate the work done in stretching the string from the equilibrium position. **(4)**

The ball is released.

c Find the speed of the ball as it passes through Q. **(3)**

d Show that in the subsequent motion the ball will not hit the ceiling. **(6)**

6 Three balls, P, Q and R, of equal radius and with masses of 3 kg, 1 kg and 2 kg respectively, lie at rest in a straight line on a smooth horizontal table. Ball P is projected towards ball Q with speed $3\,\text{m s}^{-1}$. The coefficient of restitution between each pair of balls is e. After the initial collision Q has 3.645 J of kinetic energy.

a Work out:

i the velocities of P and Q after the collision

ii the value of e. **(6)**

Q then moves on to collide with R.

b Find the kinetic energy lost in the subsequent collision. **(8)**

c State whether or not P and Q collide again. You must justify your answer. **(2)**

7 A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision, the direction of motion of S makes an angle of 60° with the wall. Immediately after the collision S has a speed of $\frac{15}{16}u\,\text{m s}^{-1}$ and its direction of motion makes an angle of α with the wall.

a Find:

i the value of α

ii the coefficient of restitution between S and the wall. **(4)**

S then moves on and collides obliquely with another smooth sphere T of equal mass and radius. Immediately before the impact T is stationary and the velocity of S makes an angle of α with the lines of centres of the two spheres, where α is the angle found in part **a i**. The coefficient of restitution between the spheres is $\frac{3}{4}$.

b Find the speeds of S and T immediately after the collision in terms of u, and the angle the velocity of each sphere makes with the line of centres. **(5)**

Answers

CHAPTER 1

Prior knowledge 1

1 $2\sqrt{17}$ N at 14° above **i**
2 **a** 5.5 m s^{-1} **b** 6 m
3 3.2 N

Exercise 1A

1 30 m s^{-1} 2 2.5 m s^{-1} 3 3 m s^{-1}
4 6.5 m s^{-1} 5 2.59 N s (2 d.p.)

Exercise 1B

1 4 m s^{-1}
2 $\frac{20}{9}$ m s^{-1}
3 4.5 m s^{-1}
4 **a** $\frac{8}{3}$ m s^{-1} **b** $\frac{8}{3}$ N s
5 **a** 1 m s^{-1} and direction unchanged
b 15 N s
6 10
7 **a** $\frac{2u}{3}$; direction reversed
b $8mu$
8 Larger 8 m s^{-1} and smaller 4 m s^{-1}
9 **a** 3 **b** $\frac{9mu}{2}$
10 **a** 3 m s^{-1} **b** 4.5
11 **a** 4 m s^{-1} in same direction
b 3 m s^{-1} in opposite direction
12 **a** 3 m s^{-1} **b** 6 kg

Challenge

$P: I = \frac{9mu_1}{4}$ $Q: I = \frac{3mu_2}{2}$

$\frac{9mu_1}{4} = \frac{3mu_2}{2}$ gives $u_1 = \frac{2u_2}{3}$

Exercise 1C

1 $(44\mathbf{i} - 24\mathbf{j})$ m s^{-1} 2 $(8\mathbf{i} + 8\mathbf{j})$ m s^{-1}
3 $(\mathbf{i} - 2\mathbf{j})$ m s^{-1} 4 $(3\mathbf{i} - 4\mathbf{j})$ m s^{-1}
5 $(18\mathbf{i} - 24\mathbf{j})$ N s, $(7\mathbf{i} - 7\mathbf{j})$ m s^{-1}
6 $(10\mathbf{i} - 5\mathbf{j})$ N s, $(25\mathbf{i} + 2\mathbf{j})$ m s^{-1}
7 $(-12\mathbf{i} - 12\mathbf{j})$ N s 8 $(-6\mathbf{i} + 4.5\mathbf{j})$ N s
9 $|\mathbf{Q}| = 30$ $\alpha = 37°$ (nearest degree)
10 $|\mathbf{Q}| = \sqrt{5} = 2.24$ (3 s.f.) $\alpha = 27°$ (nearest degree)
11 $6\sqrt{10}$ or 19.0 N s (3 s.f.)
12 $(-5\mathbf{i} + 30\mathbf{j})$ m s^{-1}
13 $\mathbf{v} = (14\mathbf{i} + 20\mathbf{j})$ m s^{-1}
14 $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ 18° (nearest degree)
15 $6\mathbf{i}$ m s^{-1}
16 $\frac{3}{7}\sqrt{2}$

Challenge

$\mathbf{I} = m(c - a)\mathbf{i} + m(d - b)\mathbf{j}$; $\tan 45 = \frac{d - b}{c - a} = 1$; $b + c = a + d$

Mixed exercise 1

1 **a** $\frac{1}{2}u = v$, direction reversed
b $6mu$
2 **a** 14 m s^{-1} **b** $\frac{35}{3}$ m s^{-1} **c** 0.75 m (2 s.f.)
d e.g. The pile driver is likely to bounce slightly so the particles will not coalesce; the pile driver is much heavier than the pile so the particles will behave as if they coalesce.
3 **a** 2000 **b** 36 m
4 **a** 1.75 m s^{-1} **b** 0.45 N s
5 **a** 2.5 m s^{-1} **b** 15 000 N s
6 **a** 0.7 m s^{-1} **b** unchanged **c** 8.25 N s
7 $\frac{4}{5}$
8 **a** 7.5 m s^{-1} **b** 11 000 (2 s.f.)
c R could be modelled as varying with speed.
9 26 N s, 23°(nearest degree)
10 $213\mathbf{j}$ m s^{-1}
11 $7\mathbf{i} + 56\mathbf{j}$
12 **a** 61 m s^{-1} **b** $(68\mathbf{i} + 23\mathbf{j})$ m s^{-1}

Challenge

Using equations for impulse
1 Q changes direction after impact:
$km(v + u) = m(u - v)$ so $k = \frac{u - v}{u + v}$
2 P changes direction after impact
$km(u - v) = m(v + u)$ so $k = \frac{u + v}{u - v}$
a k must be positive so $u > v$
b If $k = \frac{u - v}{u + v}$ then Q changes direction after impact.
If $k = \frac{u + v}{u - v}$ then P changes direction after impact.

CHAPTER 2

Prior knowledge 2

1 **a** 107 N (3 s.f.) **b** 3.22 m s^{-2} (3 s.f.)
c 40.2 m (3 s.f.)
2 0.58 (2 d.p.)

Exercise 2A

1 2.52 J 2 8.5 N 3 24.0 J (3 s.f.)
4 588 J 5 330 J 6 73.5 J
7 38.3 m (3 s.f.)
8 **a** 228 J (3 s.f.)
b Assumption that there is no frictional force between sled and ice, reasonable assumption as coefficient of friction with ice will be very low.
9 0.255 (3 s.f.)
10 64.7 J (3 s.f.)
11 23 400 J (3 s.f.)
12 281 J (3 s.f.)
13 2.48 J (3 s.f.)
14 **a** 21.7 N (3 s.f.) **b** 326 J (3 s.f.) **c** 452 J (3 s.f.)
15 0.559 (3 s.f.)
16 112 J (3 s.f.)
17 **a** 35.3 J (3 s.f.) **b** 16.5 J (3 s.f.)
c 7.19 m s^{-1} (3 s.f.)

Exercise 2B

1 **a** 33.8 J (3 s.f.) **b** 6 J **c** 500 J
d 200 J **e** 160 000 J
Order: **e**, **c**, **d**, **a**, **b**
2 **a** 44.1 J, gain **b** 8085 J, gain
c 22 050 J, loss **d** 34 104 J, loss
3 76.8 J 4 168 750 J 5 8
6 4.53 m s^{-1} (3 s.f.)
7 **a** 728 J (3 s.f.)
b No air resistance. Valid for low speeds but not large speeds.
8 **a** 11.8 J (3 s.f.) **b** 4.85 J (3 s.f.)
9 384 000 J (3 s.f.)
10 **a** 65 625 J **b** 1 837 500 J
11 **a** 20.0 m (3 s.f.)

Challenge

1 **a** K.E. = $48.0t^2$, P.E. = $-48.0t^2$, measured from the top of the cliff.
b K.E. + P.E. = $4.9t^2 - 4.9t^2 = 0$

Exercise 2C

1 a 27.4 J (3 s.f.) b $11.7\,\text{m s}^{-1}$ (3 s.f.)
2 a 36 J b 36 J c 7.35 m
3 a 56.3 J (3 s.f.) b 56.3 J (3 s.f.) c 5.63 m (3 s.f.)
4 a 9.6 J b 9.6 J c 0.350
5 a 54 J b 54 J c 4.59 m
6 $9.90\,\text{m s}^{-1}$ (3 s.f.)
7 20.4 m (3 s.f.)
8 10.6 (3 s.f.)
9 250 000 N (or 250 kN)
10 a 0.075 m (or 75 mm)
b It could depend on the speed of the bullet.
11 a 56.2 J (3 s.f.) b 56.2 J (3 s.f.)
c $4.74\,\text{m s}^{-1}$ (3 s.f.)
12 0.408 (3 s.f.)
13 8.27 (3 s.f.)
14 $7.94\,\text{m s}^{-1}$ (3 s.f.)
15 2.33 m (3 s.f.)
16 a $29.3\,\text{m s}^{-1}$ (3 s.f.) b 9.57 J (3 s.f.)
c 0.354 N (3 s.f.)
17 a 284 N
b Resistive forces could depend on speed.
18 128 N (3 s.f.)
19 11.8 m (3 s.f.)

Challenge
$1130\,\text{m s}^{-1}$ (3 s.f.)

Exercise 2D

1 18 kW
2 15 000 W (or 15 kW)
3 278 N
4 $25\,\text{m s}^{-1}$
5 a 20 000 W (or 20 kW)
b Typically resistance increases with velocity.
6 550 N
7 a $1.10\,\text{m s}^{-2}$ b $0.294\,\text{m s}^{-2}$ c $25.7\,\text{m s}^{-1}$
8 11 400 W (or 11.4 kW)
9 $R = 300$ N
10 $10\,\text{m s}^{-1}$
11 a 175 N (3 s.f.) b $0.854\,\text{m s}^{-2}$ (3 s.f.)
12 a $0.868\,\text{m s}^{-2}$ (3 s.f.) b $13.2\,\text{m s}^{-1}$ (3 s.f.)
13 a 18 000 W (or 18 kW) b $6.80\,\text{m s}^{-1}$
14 192 W
15 a $6.11\,\text{m s}^{-2}$ b $0.342\,\text{m s}^{-2}$
16 37.9
17 a $6.11\,\text{m s}^{-1}$ (3 s.f.) b $0.342\,\text{m s}^{-2}$ (3 s.f.)

Mixed exercise 2

1 20.2 N (3 s.f.)
2 a 2940 J b $98\,\text{J s}^{-1}$ (or 98 W)
3 a 20 J b 0.163
4 a $4.48\,\text{m s}^{-2}$ (3 s.f.) b 1.51 m (3 s.f.)
5 a $0.708\,\text{m s}^{-2}$ (3 s.f.) b $0.521\,\text{m s}^{-2}$ (3 s.f.)
6 a 11.4 kW (3 s.f.) b 21.3 (3 s.f.)
7 a $\dfrac{9mgs}{5}$ b $\dfrac{14gs}{25}$
8 a $2.95\,\text{m s}^{-1}$ (3 s.f.) b 61.2 J (3 s.f.)
9 $0.2\,\text{m s}^{-2}$
10 32 600 000 J (or 32 600 kJ) (3 s.f.)
11 a 16.2 J b 16.2 J c 4.05 N
12 a 250 J b 0.638 (3 s.f.)
13 a 480 N b $25.4\,\text{m s}^{-1}$ (3 s.f.)
14 $0.15\,\text{m s}^{-1}$
15 a 7.42 N (3 s.f.) b 435 J (3 s.f.)
c $14.0\,\text{m s}^{-1}$ (3 s.f.)
16 a $33.3\,\text{m s}^{-1}$ (3 s.f.)
b $0.222\,\text{m s}^{-2}$ (3 s.f.)
17 a 12 kW b 24 kW c 10.8 (3 s.f.)
18 a $9.8\,\text{ms}^{-2}$ (3 s.f.) b $31.3\,\text{ms}^{-1}$ (3 s.f.)
19 a $4.9 - k\,\text{ms}^{-2}$ b 0.98

Challenge
a $588\,000\sin\theta\,W$
b When $\theta = 0°$, no force to act against so no power required. When $\theta = 90°$, maximum power is needed.

CHAPTER 3

Prior knowledge 3

1 $F = 2\sqrt{7}$ N, $\tan\theta = \dfrac{\sqrt{3}}{2}$ 2 $\mu = \dfrac{3}{g}$
3 $PQ = 2.55$ m (3 s.f.)

Exercise 3A

1 a 3.4 m b 4 m c 3.75 m
2 4 m 3 1.31 m (3 s.f.) 4 $\dfrac{mga}{\lambda}$
5 $l = \dfrac{m_1a_2 - m_2a_1}{m_1 - m_2}$
$\lambda = g\dfrac{(m_1a_2 - m_2a_1)}{(a_1 - a_2)}$
6 $W = \frac{100}{3}$ N
7 a $\dfrac{11a}{4}$
b If the spring is not light then in effect the mass would increase, the extension would increase and hence the distance of the particle below the ceiling would increase.
8 a 14.7 N b 2.7 m (2 s.f.)
9 $\dfrac{11l}{4}$
10 a 1 m b 0.58 m (2 s.f.) c 17 N (2 s.f.)
11 a 3.9 N (2 s.f.) b 0.96 m (2 s.f.)

Exercise 3B

1 a $6\,\text{m s}^{-2}$ b $2\,\text{m s}^{-2}$
2 $12.5\,\text{m s}^{-2}$
3 $14.2\,\text{m s}^{-2}$ upwards (3 s.f.)
4 $3.13\,\text{m s}^{-2}$ downwards (3 s.f.)
5 a $4.62\,\text{m s}^{-2}$ (3 s.f.)
b Resultant force down plane $= T + g\sin\alpha - \mu R = ma$ so if μ increases acceleration would decrease.

Challenge
a Resolving vertically: $3g - 2T\cos 45 = \dfrac{3g}{2}$, gives $T = \dfrac{3\sqrt{2}\,g}{4}$
b $\lambda = 3\sqrt{2g}$

Exercise 3C

1 1.07 J (3 s.f.) 2 0.1 J 3 1.125 J
4 a 0.571 J (3 s.f.) b 1.14 J (3 s.f.) c 3.43 J (3 s.f.)
5 23 J (2 s.f.)
6 $2mga$
7 a $T = mg\dfrac{\sqrt{41}}{4}$ N b $\dfrac{41mga}{64}$ J

Exercise 3D

1 $V = \frac{1}{2}\sqrt{gl}$
2 $\dfrac{3a}{4} = d$
3 a Modulus is $mg\sqrt{3}$
b Take into account the mass of the spring.
4 a $V = 2\,\text{m s}^{-1}$ b 0.80 m (2 s.f.)
5 $U = \sqrt{\dfrac{3ag}{2}}$
6 a 160 N (2 s.f.) b $2.9\,\text{m s}^{-1}$ (2 s.f.)
7 a Falls 4 m b $6.6\,\text{m s}^{-1}$ (2 s.f.)
8 0.11 (2 s.f.)

Online Full worked solutions are available in SolutionBank.

Challenge

When extension is $\frac{l}{10}$ E.P.E. $= \frac{Mgl}{20}$

When extension is doubled, E.P.E. $= \frac{4Mgl}{20}$

Work done is difference $= \frac{3Mgl}{20}$

Mixed exercise 3

1 a If $\cos\theta = \frac{4}{5}$ then $\sin\theta = \frac{3}{5}$

Resolving vertically: $2T\cos\theta = mg$, gives $T = \frac{5mg}{8}$

Using trigonometry: $\sin\theta = \frac{3}{5} = \frac{a}{a+x}$ gives $x = \frac{2a}{3}$

Substituting into $T = \frac{\lambda x}{a} = \frac{5mg}{8}$ gives $\lambda = \frac{15mg}{16}$

so $\cos\theta = \frac{4}{5}$

b $\frac{11mga}{12}$

2 $3a$

3 a $\lambda = 30$ N b $v = 2.19$ m s^{-1} (3 s.f.)

4 $l = \frac{5\lambda a}{(5\lambda + 3mg)}$

5 a $V = \sqrt{\frac{13ag}{20}}$ b $d = \frac{13a}{50}$

6 a $\mu = \frac{2}{3}$ b $V = \sqrt{\frac{2gl}{3}}$ c $\frac{3l}{2}$

7 a extension of AP $(x_1) = 0.2\cos\theta - 0.15$;
extension of BP $(x_2) = 0.2\sin\theta - 0.05$;

$\Rightarrow$ ratio is $\frac{0.2\cos\theta - 0.15}{0.2\sin\theta - 0.05} = \frac{4\cos\theta - 3}{4\sin\theta - 1}$

b $T_2 = 5g\cos\theta$, $T_1 = 5g\sin\theta \Rightarrow \frac{T_2}{T_1} = \frac{\cos\theta}{\sin\theta}$

$\frac{\lambda x_2}{0.05} \times \frac{0.15}{\lambda x_1} = \frac{\cos\theta}{\sin\theta} \Rightarrow \frac{x_1}{x_2} = \frac{3\sin\theta}{\cos\theta}$

Using answer to part **a**, $\frac{4\cos\theta - 3}{4\sin\theta - 1} = \frac{3\sin\theta}{\cos\theta}$

Rearrange to arrive at required solution.

8 a $\theta = \tan^{-1}\left(\frac{1}{3}\right)$ b 2.1 m (2 s.f.) c 9.3 N (2 s.f.)

9 a When AP is vertical, $x = 4a$
K.E. gain + E.P.E. gain = P.E. loss

$\Rightarrow \frac{1}{2}mv^2 + \frac{mg}{4}\frac{x^2}{2a} = mg4a$

$\Rightarrow v = 2\sqrt{ga}$

b $T = mg$

Challenge

a If x is maximum distance, using conservation of energy
$mgx = \frac{\lambda(x-l)^2}{2l}$
Expanding and rearranging to form a quadratic in x
$\lambda x^2 - x(2l\lambda + 2mgl) + l^2\lambda = 0$
Use quadratic formula to arrive at required solution.

b i For a greater maximum descent the model could include an initial velocity i.e. the person could jump rather than fall.

ii For a smaller maximum descent air resistance could be incorporated in to the model.

Review exercise 1

1 6.3 N s

2 a 16 m
b Air resistance would result in a greater deceleration.

3 a 2.25 m s^{-1}, direction unchanged
b 1.5 N s

4 a 2.4 m s^{-1} b Direction reversed
c 3000 kg

5 a A: 2.2 m s^{-1}; B: 3 m s^{-1}
b $mu = 1.5$ N s, $mv = 1.1$ N s
$mu - mv = 0.4$ N s
c 1.6 N s

6 3 m s^{-1}

7 a 3.6 kg b 18 N s

8 a $\mathbf{v} = 10\mathbf{i} + 20\mathbf{j}$ m s^{-1} b 63.4° (3 s.f.)
c 40 J

9 a 7.5 N b $\mathbf{v} = 39\mathbf{i} - 42\mathbf{j}$ m s^{-1}

10 a 5.83 N s (3 s.f.) b 31°
c 35 J

11 a $v = (2t + 2)\mathbf{i} + (3t^2 - 4)\mathbf{j}$ b 13 m s^{-1}

12 a 610 N (3 s.f.) b 458 kJ (3 s.f.)
c 801 kJ (3 s.f.)

13 Work done $= mgh \Rightarrow 19600 = 1000 \times 9.8 \times 25\sin\theta$

$\Rightarrow \sin\theta = \frac{19\,600}{1000 \times 9.8 \times 25} = \frac{2}{25} \Rightarrow \theta = \arcsin\left(\frac{2}{25}\right)$

14 a 1568 J b It will be less than 1568 J.

15 a 175 J b 196 kJ

16 a 8.4 m s^{-1} (3 s.f.) b $\mu = 0.42$ (2 s.f.)

17 a 41 J (2 s.f.) b $\mu = 0.67$ (2 s.f.)

18 a 22.4 J b 6.4 m s^{-1} (2 s.f.)
c 4.3 m s^{-1} (2 s.f.)

19 a $\frac{7}{5}mgh$ b $\frac{3}{5}gh$

20 a $50 = F \times 25 \Rightarrow F = 2000$
For the car and trailer combined:
R($\rightarrow$): $F - 750 - R = 0$
$R = F - 750 = 2000 - 750 = 1250$
b 1.4 m s^{-2} c 850 N d 335 kJ
e The resistance could be modelled as varying with speed.

21 a 0.8 b $\frac{245}{3}$
c Resistance usually varies with speed.

22 a Energy lost = 7832 J; Work done = 10 000 J
Total work done = 2168 J = 2200 J (2 s.f.)
b 200 W

23 220

24 a 0.7 m s^{-2} b 44.4 kW

25 a 35 m s^{-1} b 15 m s^{-1} (2 s.f.)

26 a Let F N be the magnitude of the driving force produced by the engine of the car.
12 kW $= F \times 15 \Rightarrow F = 800$
R($\rightarrow$): $\mathbf{F} = m\mathbf{a}$
$F - R = 1000 \times 0.2 \Rightarrow R = 600$
b 20 (2 s.f.)

27 a 14.3 kW (3 s.f.) b 6.6 kW (2 s.f.)

28 a 0.15 m s^{-2} b 35.1 m s^{-1} (3 s.f.)

29 a K.E. gained = P.E. lost

$\frac{1}{2}\cancel{m}v^2 - \frac{1}{2}\cancel{m}u^2 = \cancel{m}gh$

$\frac{1}{2} \times 24.5^2 - \frac{1}{2}u^2 = 9.8 \times 15$

$u^2 = 24.5^2 - 2 \times 9.8 \times 15 = 306.25$

$u = \sqrt{306.25} = 17.5$ (3 s.f.)

b 55° (nearest degree) c 60 m

30 a 52 (2 s.f.) b 3 s
c 48 m d 24 m s^{-1} (2 s.f.)

31 a 1.05 m b 7.5 N

32 $AB = 2.55$ m, $BC = 1.45$ m

33 $\lambda = 42$

34 a The line of action of the weight must pass through C which is not above the centre of the rod.
b Let the tension in AC be T_1 newtons and the tension in BC be T_2 newtons.
R($\uparrow$): $T_1\cos\alpha = T_2\sin\alpha \Rightarrow T_1 = \frac{3}{4}T_2$
R($\rightarrow$): $T_1\sin\alpha + T_2\cos\alpha = 2mg$
$\frac{3}{4}T_2 \times \frac{3}{5} + T_2 \times \frac{4}{5} = 2mg \Rightarrow T_2 = \frac{8}{5}mg$
c $k = 8$

35 $a = 11\,\text{m s}^{-1}$ (2 s.f.)

36 $\lambda = \frac{15}{16}mg$

37 **a** $AC = 4\,\text{m}$
b The instantaneous acceleration of B at C is $29.4\,\text{m s}^{-1}$ directed towards A.

38 **a** $\frac{5\sqrt{3}}{3}\,\text{m s}^{-1} \approx 2.89\,\text{m s}^{-1}$ (3 s.f.)
b 7.79 N (3 s.f.)

39 **a** $AP^2 = (1.5l)^2 + (2l)^2 = 6.35l^2 \Rightarrow AP = 2.5l$
Let α be the angle between AP and the vertical
$\cos\alpha = \frac{2l}{2.5l} = \frac{4}{5}$
The extension of half of the string, AP, is $2.5l - 1.5l = l$

$$T = \frac{\lambda \times \text{extension}}{\text{natural length}} = \frac{\lambda l}{1.5l} = \frac{2\lambda}{3} \quad (1)$$

R(↑): $2T\cos\alpha = mg$

$$2T \times \tfrac{4}{5} = mg \Rightarrow T = \frac{5mg}{8} \quad (2)$$

Eliminating T between (1) and (2)

$$\lambda = \frac{5mg}{8} \times \frac{3}{2} = \frac{15mg}{16}$$

b Let the perpendicular distance from the original position of P to AB be h.
$h^2 = (3.9l)^2 - (1.5l)^2 = 12.96l^2 \Rightarrow h = 3.6l$
Let the speed of P as it reaches AB be $v\,\text{m s}^{-1}$
K.E. gained + P.E. gained = E.P.E lost

$$\tfrac{1}{2}mv^2 + mg \times 3.6l = 2 \times \frac{\left(\frac{15mg}{16}\right)(3.9l - 1.5l)^2}{2 \times 1.5l}$$

$$\tfrac{1}{2}mv^2 + 3.6mgl = \frac{5mg}{8l} \times (2.4l)^2 = 3.6mgl$$

Hence $\frac{1}{2}mv^2 = 0 \Rightarrow v = 0$
P comes to instantaneous rest on the line AB.

40 By work–energy principle:
E.P.E. = Work done $= \frac{\lambda x^2}{2l}$
From Hooke's law:
$T = \frac{\lambda x}{l} = mg \Rightarrow x = \frac{mgl}{\lambda}$
E.P.E. $= \frac{\lambda m^2g^2l^2}{2l\lambda^2} = \frac{m^2g^2l}{2\lambda}$

41 2.85 J (3 s.f.)

42 **a** 14 (2 s.f.) **b** 0.78 m (2 s.f.) **c** 6.1 J (2 s.f.)

43 **a** $0.2mga^2$ **b** $\mu = 0.6$

44 $\frac{3}{4}a$

45 $\frac{1}{2}mv^2 = \frac{\lambda x^2}{2l} \Rightarrow 2.5v^2 = \frac{75 \times 0.5^2}{2 \times 1} \Rightarrow v = \frac{\sqrt{15}}{2}$

46 **a** $5.5\,\text{m s}^{-1}$ (2 s.f.) **b** $3.8\,\text{m s}^{-1}$ (2 s.f.)

47 **a** 6 m **b** $14\,\text{m s}^{-1}$ (2 s.f.)

48 **a** $1.8\,\text{m s}^{-1}$ (2 s.f.)
b Let $AP = y$ m and the angle AP makes with the vertical be α.
$\sin\alpha = \frac{0.75}{y} \Rightarrow y = \frac{0.75}{\sin\alpha}$
R(↑): $2T\cos\alpha = 2g \Rightarrow T = \frac{g}{\cos\alpha} = \frac{9.8}{\cos\alpha}$

Hooke's law:
$T = \frac{\lambda x}{l} = \frac{49}{0.75}(y - 0.75)$
$= \frac{49}{0.75}\left(\frac{0.75}{\sin\alpha} - 0.75\right)$
$= 49\left(\frac{1 - \sin\alpha}{\sin\alpha}\right)$
$\frac{9.8}{\cos\alpha} = 49\left(\frac{1 - \sin\alpha}{\sin\alpha}\right)$
$\frac{\sin\alpha}{\cos\alpha} = \frac{49}{9.8}(1 - \sin\alpha) = 5(1 - \sin\alpha)$
$\tan\alpha = 5 - 5\sin\alpha \Rightarrow \tan\alpha + 5\sin\alpha = 5$

Challenge

1 **a** Over large changes in distance from a large mass g is not constant. The actual value of work done will be less.
b Work done $= \left(3.99 \times 10^{14} \times \frac{420\,000}{6380 \times 10^3}\right)$
$- \left(3.99 \times 10^{14} \times \frac{420\,000}{6380 \times 10^3 + 405 \times 10^3}\right)$
$= 1\,567\,857\,828\,000$ J
$= 1.57 \times 10^{12}$ J

2 **a** Work done $= \int_0^x T\,\text{d}s$
$= \int_0^x \frac{\lambda s}{l}\,\text{d}s$
$= \left[\frac{\lambda s^2}{2l}\right]_0^x$
$= \frac{\lambda x^2}{2l}$
b Work done = E.P.E. gain of string
$= \frac{\lambda}{2l}(b^2 - a^2)$
$= \frac{\lambda}{2l}(b + a)(b - a)$
$= \frac{1}{2}\left(\frac{\lambda b}{l} + \frac{\lambda a}{l}\right)(b - a)$
$= \frac{1}{2}(T_b + T_a)(b - a)$
= mean of tensions × distance moved

3 1.24 m (3 s.f.)

CHAPTER 4

Prior knowledge 4

1 **a** $1.25\,\text{m s}^{-1}$ direction reversed.
b 0.9 N s

2 $25\,\text{m s}^{-1}$

3 31.9 m (3 s.f.)

Exercise 4A

1 **a** $\frac{2}{3}$ **b** $\frac{1}{2}$ **c** $\frac{1}{3}$

2 **a** $v_1 = 0, v_2 = 3$
b $v_1 = 2.5, v_2 = 3$
c $v_1 = 4, v_2 = 6$
d $v_1 = -4, v_2 = 4$
e $v_1 = -5, v_2 = -2$

3 **a** $3.5\,\text{ms}^{-1}$ **b** 1

4 $5\,\text{m s}^{-1}$ and $3\,\text{m s}^{-1}$ both in the direction that B was moving before the impact.
18 N s

5 $\frac{u}{2}$ direction reversed, $\frac{1}{4}$

6 $\frac{u}{3}(5 - 4e), \frac{u}{3}(5 + 2e)$

7 Using conservation of linear momentum gives
$2mu - 3mu = -mv_1 + mv_2 \Rightarrow -v_1 + v_2 = -u \quad (1)$
Newton's law of restitution gives $e = \frac{v_2 + v_1}{5u} \quad (2)$
Eliminating v_1 from (1) and (2) gives
$v_2 = \frac{1}{2}u(5e - 1)$
Since v_2 is in the positive direction $v_2 > 0$, so $\frac{1}{2}u(5e - 1) > 0$, and $e > \frac{1}{5}$

8 a $u\left(1 - \frac{3k}{10}\right)$

b Newton's law of restitution gives

$eu = \frac{3}{10}u - u\left(1 - \frac{3k}{10}\right)$, so $e = \frac{1}{10}(3k - 7)$

Since $0 \leqslant e \leqslant 1$,

$0 \leqslant \frac{1}{10}(3k - 7) \leqslant 1 \Rightarrow \frac{7}{3} \leqslant k \leqslant \frac{17}{3}$

9 a $u(5 - 3k)$

b Newton's law of restitution gives

$eu = ku - u(5 - 3k)$, so $e = 4k - 5$

Since $0 \leqslant e \leqslant 1$, $0 \leqslant 4k - 5 \leqslant 1$

$\Rightarrow \frac{5}{4} \leqslant k \leqslant \frac{3}{2}$

10 a Using conservation of linear momentum gives

$4mu + 6mu = mv_1 + 3mv_2 \Rightarrow v_1 + 3v_2 = 10u$ (1)

Newton's law of restitution gives

$e = \frac{v_2 - v_1}{2u}$ (2)

Eliminating v_1 from (1) and (2) gives

$v_2 = \frac{u}{2}(5 + e)$

b $\frac{u}{2}(5 - 3e)$

c After the collision P has speed $\frac{u}{2}(5 + 3e)$ which is always positive, therefore P continues to move in the same direction.

d $e = \frac{1}{3}$

Challenge

Using conservation of linear momentum gives

$6m - mu = 3mv + 2mv \Rightarrow 6 - u = 5v$

so $v = \frac{1}{5}(6 - u)$ (1)

Newton's law of restitution gives

$\frac{1}{4} = \frac{v}{2 + u}$ so $v = \frac{1}{4}(2 + u)$ (2)

Eliminating v from (1) and (2) gives $u = \frac{14}{9}$

Exercise 4B

1 a $\frac{2}{5}$ **b** $\frac{1}{2}$

2 a $3.5\,\text{m s}^{-1}$ **b** $3\,\text{m s}^{-1}$

3 a $8\,\text{m s}^{-1}$ **b** $8\,\text{m s}^{-1}$

4 $e = 0.75$

5 0.77 (2 s.f.)

6 a 0.1875 **b** Particle would rebound higher.

7 $\frac{1}{2}$ **8** 2.94 s **9** $\left(\sqrt{gh} - \frac{1}{2}g\right)$m

Challenge

Use $v^2 = u^2 + 2as$ with $u = 0\,\text{m s}^{-1}$, $a = g\,\text{m s}^{-2}$ and $s = h\,\text{m}$

$v = \sqrt{2gh}$

Newton's law of restitution gives

speed of separation from floor $= e\sqrt{2gh}\,\text{m s}^{-1}$

Use $v^2 = u^2 + 2as$ with $u = e\sqrt{2gh}\,\text{m s}^{-1}$, $a = -g\,\text{m s}^{-2}$ and $v = 0\,\text{m s}^{-1} \Rightarrow s = he^2$

Exercise 4C

1 a $6\,\text{m s}^{-1}$ **b** 1.5 J

2 A has speed $\frac{7u}{3}$ with direction of travel reversed and B has speed $\frac{u}{3}$ towards A.

Loss of K.E. is $\frac{5mu^2}{3}$

3 60 J

4 0.225 J

5 a $2\,\text{m s}^{-1}$ **b** 12 060 J (12.06 kJ)

6 a $\frac{5}{3}\,\text{m s}^{-1}$ **b** $1606\frac{2}{3}$ J

7 a $N = 3$ **b** $\frac{3}{8}$

8 a $0.3\,\text{m s}^{-1}$ **b** $\frac{1}{5}$ **c** 3600 J

9 $v - u$ and $v + \frac{1}{2}u$

10 a Conservation of linear momentum:

$8 + 3 = 2u + 3v$

$\Rightarrow 11 = 2u + 3v \Rightarrow \frac{11 - 3v}{2} = u$

Kinetic energy:

$16 + 1.5 - 3 = u^2 + 1.5v^2$

Substitute for u:

$\Rightarrow \left(\frac{11 - 3v}{2}\right)^2 + 1.5v^2 = 14.5$

$\Rightarrow 3.75v^2 - \frac{33}{2}v + \frac{121}{4} = 14.5$

$\Rightarrow 5v^2 - 22v + 21 = 0$

b A moves with speed $1\,\text{m s}^{-1}$ and B moves with speed $3\,\text{m s}^{-1}$ in the same direction as before the impact. A cannot travel faster than B as it cannot pass through it.

11 a $2\,\text{m s}^{-1}$ **b** 35 J

12 Common speed before string becomes taut $= \frac{mu}{M + m}$

Kinetic energy before collision $= \frac{1}{2}mu^2$

Kinetic energy after collision $= \frac{1}{2}(m + M)\left(\frac{mu}{m + M}\right)^2$

$= \frac{m^2u^2}{2(m + M)}$

Loss of kinetic energy $= \frac{1}{2}mu^2 - \frac{m^2u^2}{2(m + M)} = \frac{mMu^2}{2(m + M)}$

13 a $7.5\,\text{m s}^{-1}$ **b** 375 J

14 a $0.32\,\text{s}$, $2.5\,\text{m s}^{-1}$ **b** 0.375 J

Challenge

2.5 J

Exercise 4D

1 a $u = 3, v = 5, x = 4, y = 4.5$

b $u = 2, v = 4, x = 3.5, y = 4$

2 $-1.5\,\text{m s}^{-1}$, $0.5\,\text{m s}^{-1}$, $5\,\text{m s}^{-1}$,

3 a $\frac{1}{2}u(1 - e)$, $\frac{1}{4}u(1 + e)(1 - e)$ and $\frac{1}{4}u(1 + e)^2$

b A will catch up with B provided that $2 > 1 + e$. Since $e < 1$ this condition holds and A will catch up with B, resulting in a further collision.

4 a $u(1 + 3e) > 3u \Rightarrow e > \frac{2}{3}$

b The direction of A is reversed by the collision.

5 $\frac{5u}{8}$ and $\frac{7u}{12}$

6 a $-3u$ and $5u$ **b** $\frac{17u}{4}$ and $\frac{43u}{12}$

7 a i 19.6 cm **ii** 9.604 cm

b A series of bounces with a constant ratio of heights.

c 1.17 m (3 s.f.)

d Model predicts an infinite number of bounces, this is unrealistic.

8 a e^2H **b** e^4H **c** $\frac{H(1 + e^2)}{(1 - e^2)}$

9 $\frac{d}{2}\left(\frac{1}{2} + \frac{1}{e_1} + \frac{1}{e_1e_2}\right)$

Challenge

After the particles collide: $v(P) = 0.25\,\text{m s}^{-1}$, $v(Q) = 1.25\,\text{m s}^{-1}$

t = time in seconds after the particles collide.

Q collides with W_2 after travelling 2 m, i.e. after $t = 1.6$ s.

P will take $t = 8$ s to reach W_1.

After Q collides with W_2, velocity $= 0.64\,\text{m s}^{-1}$.

For particle P, distance from W_2 at time $t = 2 + 0.25t$

For particle Q, distance from W_2 at time $t = 0.64(t - 1.6)$

P and Q will collide when $2 + 0.25t = 0.64(t - 1.6) \Rightarrow t = 8.4$ s

Time to collide > time for P to reach W_1.

P will hit W_1 before colliding with Q for a second time.

Mixed exercise 4

1 Using conservation of linear momentum gives
$mu_1 - mu_2 = mv$
$u_1 - u_2 = v$ (1)
Newton's law of restitution gives
$\frac{1}{3} = \frac{v}{u_1 + u_2}$ so $u_1 + u_2 = 3v$ (2)
Solving (1) and (2) gives
$u_1 = 2v$ and $u_2 = v$
So the ratio of speeds is 2:1

2 4

3 a $V = \frac{mv}{M}$

b Kinetic energy $= \frac{mv^2}{2} + \frac{MV^2}{2}$
Substitute for V: $\frac{mv^2}{2} + \frac{m^2v^2}{2M} = \frac{m(m + M)v^2}{2M}$

c The assumption that the boat is initially at rest is unlikely.

4 $2.5\,\text{m s}^{-1}$, 42.5 J

5 a Using conservation of linear momentum gives
$3mu = 3mv_1 + mv_2$
$3v_1 + v_2 = 3u$ (1)
Newton's law of restitution gives
$v_2 - v_1 = eu$ (2)
Eliminating v_2 from (1) and (2) gives
$v_1 = \frac{u}{4}(3 - e)$

b $v_2 = \frac{3u}{4}(e + 1)$
Kinetic energy before collision $= \frac{3}{2}mu^2$
Kinetic energy after collision
$= \frac{3mu^2}{32}(3 - e)^2 + \frac{9mu^2}{32}(e + 1)^2$
Loss of kinetic energy
$= \frac{3}{2}mu^2 - \left(\frac{3mu^2}{32}(3 - e)^2 + \frac{9mu^2}{32}(e + 1)^2\right)$
$= \frac{3}{2}mu^2 - \frac{mu^2}{32}[3(3 - e)^2 + 9(e + 1)^2]$
$= \frac{3}{2}mu^2 - \frac{mu^2}{32}[3(9 - 6e + e^2) + 9(e^2 + 2e + 1)]$
$= \frac{3}{2}mu^2 - \frac{mu^2}{32}(12e^2 + 36)$
$= \frac{3}{2}mu^2 - \frac{3mu^2}{8}(e^2 + 3)$
$= \frac{12}{8}mu^2 - \frac{3mu^2}{8}(e^2 + 3)$
$= \frac{12}{8}mu^2 - \frac{3}{8}mu^2e^2 + \frac{9}{8}mu^2$
$= \frac{3}{8}mu^2 - \frac{3}{8}mu^2e^2$
$= \frac{3}{8}mu^2(1 - e^2)$

c $\frac{3mu(1 + e)}{4}$ N s

6 a $6\,\text{m s}^{-1}$ and $1\,\text{m s}^{-1}$ in the direction of the 100 g mass prior to the impact.
b Loss of K.E. = 2.45 J

7 3 s after the 10 kg sphere has stopped moving.

8 First collision: A with B
$4mV = 4mv_1 + 3mv_2$
$\frac{v_2 - v_1}{V} = \frac{3}{4} \Rightarrow v_2 = \frac{3V}{4} + v_1$
$4V = 4v_1 + \frac{9V}{4} + 3v_1$
$4V = 7v_1 + \frac{9V}{4}$
$v_1 = \frac{1}{4}V, v_2 = V$
$v_2 > v_1$ so second collision is B with C
$3mV = 3mv_3 + 3mv_4$
$\frac{v_4 - v_3}{V} = \frac{3}{4} \Rightarrow v_4 = \frac{3V}{4} + v_3$
$3V = 3v_3 + \frac{9V}{4} + 3v_3$
$3V = 6v_3 + \frac{9V}{4}$
$v_3 = \frac{1}{8}V, v_4 = \frac{7}{8}V$
$v_4 > v_3$ but $v_1 > v_3$ so third collision is A with B
$mV + \frac{3}{8}mV = 4mv_5 + 3mv_6$
$\frac{v_6 - v_5}{\frac{1}{4}V - \frac{1}{8}V} = \frac{3}{4} \Rightarrow v_6 = \frac{3}{32}V + v_5$
$\frac{11}{8}V = 4v_5 + \frac{9}{32}V + 3v_5$
$\frac{11}{8}V = 7v_5 + \frac{9}{32}V$
$v_5 = \frac{5}{32}V, v_6 = \frac{1}{4}V$
As $\frac{5}{32}V < \frac{1}{4}V < \frac{7}{8}V$ there are no further collisions.

9 a 9072 J b *Either* heat *or* sound.

10 a $\frac{u}{7}(4 - 3e)$ and $\frac{4}{7}u(1 + e)$
b Impulse = change in momentum
$2mu = \frac{12mu}{7}(1 + e)$
$1 + e = \frac{7}{6}$
$e = \frac{1}{6}$

11 a Using conservation of linear momentum gives
$mkV + \lambda mV = \lambda mv$ so $V = \frac{\lambda v}{\lambda + k}$ (1)
Newton's law of restitution gives
$e = \frac{v}{V(k - 1)}$ (2)
Eliminating V from (1) and (2) gives
$e = \frac{\lambda + k}{\lambda(k - 1)}$

b $\frac{\lambda + k}{\lambda(k - 1)} < 1$
$\lambda + k < \lambda(k - 1)$
$\lambda + k < \lambda k - \lambda$
$2\lambda - \lambda k < -k$
$\lambda k - 2\lambda > k$
$\lambda(k - 2) > k$
$\lambda > \frac{k}{k - 2}$ and since $\lambda > 0$, $k - 2 > 0$, $k > 2$

12 a Both balls change directions, the first moves up with speed $0.7\,\text{m s}^{-1}$ and the second moves down with speed $3.5\,\text{m s}^{-1}$.
b 91% (2 s.f.)

13 a 0.5 m
b $\frac{2}{\sqrt{g}}$ s or 0.64 s (2 s.f.)
c $\frac{1}{4}\sqrt{g}\,\text{m s}^{-1}$ or $0.78\,\text{m s}^{-1}$ (2 s.f.)

14 To find time taken for ball to make first contact with floor use
$s = ut + \frac{1}{2}at^2$ with $u = 0\,\text{m s}^{-1}$, $s = h\,\text{m}$ and $a = g\,\text{m s}^{-2}$
$t = \sqrt{\frac{2h}{g}}$
To find speed of ball on first contact with floor use
$v^2 = u^2 + 2as$ with $u = 0\,\text{m s}^{-1}$, $s = h\,\text{m}$ and $a = g\,\text{m s}^{-2}$
$v = \sqrt{2gh}$
Newton's law of restitution gives
speed of separation from floor $= e\sqrt{2gh}$
To find time between first contact and maximum height reached $v = u + at$ with $u = e\sqrt{2gh}\,\text{m s}^{-1}$, $v = 0\,\text{m s}^{-1}$ and $a = -g\,\text{m s}^{-2}$

$t = e\sqrt{\frac{2h}{g}}$

So the time between the first and second bounce is $2e\sqrt{\frac{2h}{g}}$

By similar reasoning the time between the second and third bounce is $2e^2\sqrt{\frac{2h}{g}}$ and the time between the third and fourth bounce is $2e^3\sqrt{\frac{2h}{g}}$

So the total time until the ball comes to a standstill is given by

$$t = \sqrt{\frac{2h}{g}} + 2e\sqrt{\frac{2h}{g}} + 2e^2\sqrt{\frac{2h}{g}} + 2e^3\sqrt{\frac{2h}{g}} + \ldots$$
$$= \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2h}{g}}(e + e^2 + e^3 + \ldots)$$
$$= \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2h}{g}}\left(\frac{e}{1-e}\right)$$
$$= \sqrt{\frac{2h}{g}}\left(\frac{1-e}{1-e}\right) + 2\sqrt{\frac{2h}{g}}\left(\frac{e}{1-e}\right)$$
$$= \frac{1+e}{1-e}\sqrt{\frac{2h}{g}}$$

15 $\frac{25}{32}$

16 $V = \sqrt{\frac{2ME}{m(M+m)}}\ \mathrm{m\,s^{-1}}$

17 **a** Use $v^2 = u^2 + 2as$ with $u = 0\,\mathrm{m\,s^{-1}}$, $a = g\,\mathrm{m\,s^{-2}}$ and $s = H\,\mathrm{m}$
$v = \sqrt{2gH}$
Newton's law of restitution gives
speed of separation from floor $= e\sqrt{2gH}\,\mathrm{m\,s^{-1}}$
Use $v^2 = u^2 + 2as$ with $u = e\sqrt{2gH}\,\mathrm{m\,s^{-1}}$, $a = -g\,\mathrm{m\,s^{-2}}$ and $v = 0\,\mathrm{m\,s^{-1}}$ and $s = h\,\mathrm{m}$
$2gh = 2e^2gH$
$e = \sqrt{\frac{h}{H}}$

b $\frac{h^2}{H}$

c Infinite bounces with a constant ratio of heights.

18 **a** B has speed $\frac{5}{12}\sqrt{2g}\,\mathrm{m\,s^{-1}}$ in the direction it was originally moving.
C has speed $\frac{7}{6}\sqrt{2g}\,\mathrm{m\,s^{-1}}$ in the direction B was originally moving.

b $\frac{7g}{24}$ J

c The amount of kinetic energy lost would increase.

19 6:19

Challenge

Using conservation of momentum, with common final velocity v:

$m_3u = (m_1 + m_2 + m_3)v$

So $V = \frac{m_3u}{(m_1 + m_2 + m_3)}$

Kinetic energy when all three strings are taut

$= \frac{1}{2}(m_1 + m_2 + m_3)\left(\frac{m_3u}{m_1 + m_2 + m_3}\right)^2 = \frac{m_3^2u^2}{2(m_1 + m_2 + m_3)}$

CHAPTER 5

Prior knowledge 5

1 0.75

2 **a** $1\,\mathrm{ms^{-1}}$
b 4.5J lost

Exercise 5A

1 **a** $v = \frac{u\sqrt{17}}{5}$
b Angle of deflection = 50.9° (3 s.f.)

2 $e = \frac{1}{4}$

3 $v = \frac{3\sqrt{17}u}{13}$

4 $e = \frac{\sqrt{29}}{8}$

5 $\sqrt{19}\,\mathrm{m\,s^{-1}}$

6 $5.08\,\mathrm{m\,s^{-1}}$ (3 s.f.)

7 **a** $5.59\,\mathrm{m\,s^{-1}}$ **b** 5.625 N s

8 $e = 0.36$

9 **a** $v = -2.5\mathbf{i} - 3\mathbf{j}$ **b** 7.5 J
c Angle of deflection = 98.8° (3 s.f.)

10 **a** $\sqrt{13}\,\mathrm{m\,s^{-1}}$ **b** 16 J

11 **a** $\frac{8}{3}\mathbf{i} + \frac{10}{3}\mathbf{j}$ **b** 8.89% (3 s.f.)
c 24.8° (3 s.f.)

12 $6.09\,\mathrm{m\,s^{-1}}$ (3 s.f.) at 21.8° (3 s.f.) to the cushion

13 **a** Parallel to cushion $v\cos\beta = u\cos\alpha$
Perpendicular to cushion $v\cos\beta = eu\cos\alpha$
Dividing yields: $\tan\beta = e\tan\alpha$
Angle is independent of u
b 0.70 (2 s.f.)

14 **a** $\frac{56}{225}$ **b** $\frac{5}{9}$

15 **a** $\frac{1}{5}(-3\mathbf{i} + 4\mathbf{j})$ **b** $e = \frac{2}{23}$

16 **a** $2\sqrt{17}$ N s in the direction parallel to the unit vector $\frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j})$
b $e = \frac{7}{10}$ **c** 3 J

17 **a** By principle of conservation of momentum
$mv\cos(90° - \alpha) = 3mv\cos\alpha$
$\sin\alpha = 3\cos\alpha$
$\tan\alpha = 3$
b $\frac{1}{9}$

18 48.2° (3 s.f.)

Challenge

Let β be the angle between W_1 and the ball after the impact.
Parallel to the plane $v\cos\beta = u\cos\alpha$
Perpendicular to plane $v\sin\beta = eu\sin\alpha$
Dividing yields: $\tan\beta = e\tan\alpha$
Distance between join of walls and Q is $1 \times \tan\beta = e\tan\alpha$
From Pythagoras distance $PQ = \sqrt{e^2\tan^2\alpha + 1}$

Exercise 5B

1 **a** $1.80\,\mathrm{ms^{-1}}$ (3 s.f.), 16.1° (3 s.f.) to the wall
b $1\,\mathrm{ms^{-1}}$, 60° to the wall

2 **a** $0.815\,\mathrm{ms^{-1}}$ (3 s.f.) **b** 0.434 (3 s.f.)
c $0.434\,\mathrm{ms^{-1}}$ (3 s.f.), 50° to the second wall

3 **a** 19.7° (3 s.f.) to the wall
b 0.621 (3 s.f.)
c 0.00121 J (3 s.f.)

4 **a** $2.46\,\mathrm{ms^{-1}}$ (3 s.f.), 52.4° (3 s.f.) to the wall
b 4.61 J (3 s.f.)

5 eu, parallel to the original path but in the opposite direction.

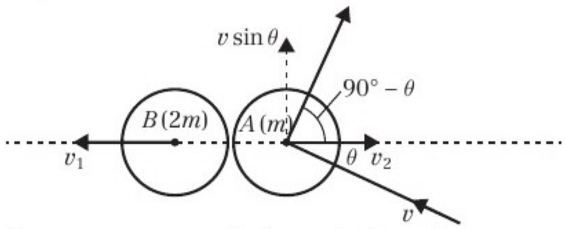
6 $\frac{\sqrt{3}u}{3}\,\mathrm{m\,s^{-1}}$

7 **a** $4.77\,\mathrm{ms^{-1}}$ (3 s.f.) at an angle of 24.8° (3 s.f.) to the first wall.
b $3.94\,\mathrm{ms^{-1}}$ (3 s.f.) at an angle of 65.3° (3 s.f.) to the second wall.
c The velocity of the sphere would be greater and the angle with the wall would be greater.

8 a $\frac{123}{8}$ mJ
b 180°, direction of motion reversed
9 a $2.06\,\text{m s}^{-1}$ (3 s.f.) at an angle of 31.0° (3 s.f.) to the wall
b 0.180 J (3 s.f.)
10 77.2 J (3 s.f.)

Challenge
a i 3.98 m ii 0.36 m
b Distance from wall would be greater, height of bounce would be greater.

Exercise 5C

1 A: $1.04\,\text{m s}^{-1}$ (3 s.f.) perpendicular to the line of centres
B: $2.95\,\text{m s}^{-1}$ (3 s.f.) parallel to the line of centres
2 $\frac{4\sqrt{39}}{9}\,\text{m s}^{-1}$ at 46.1° (3 s.f.) to the line of centres
$\frac{16\sqrt{3}}{9}\,\text{m s}^{-1}$ along the line of centres
3 $\frac{25}{7}\,\text{m s}^{-1}$ at 81.9° (3 s.f.) to the line of centres
$\frac{45\sqrt{2}}{28}\,\text{m s}^{-1}$ along the line of centres
4


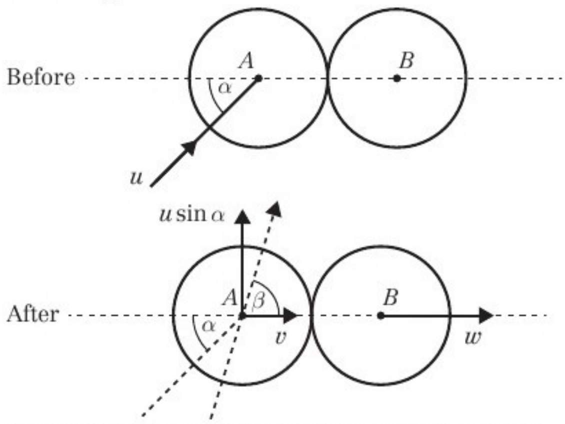
Components perpendicular to the line of centres are unchanged. For A, the component perpendicular to the line of centres is $v\sin\theta$.
Parallel to the line of centres:
conservation of momentum $\Rightarrow mv\cos\theta = 2mv_1 - mv_2$
law of restitution $\Rightarrow v_1 + v_2 = ev\cos\theta$

$$\Rightarrow v\cos\theta = 2(ev\cos\theta - v_2) - v_2 = 2ev\cos\theta - 3v_2$$

$$v_2 = \frac{v\cos\theta(2e-1)}{3}$$

$$\Rightarrow \tan(90° - \theta) = \frac{1}{\tan\theta} = \frac{v\sin\theta}{v_2} = \frac{3v\sin\theta}{v\cos\theta(2e-1)},$$

$$\therefore \frac{1}{\tan\theta} = \frac{3\tan\theta}{2e-1} \Rightarrow \tan^2\theta = \frac{2e-1}{3}$$

5 $\frac{\sqrt{61}v}{6}\,\text{m s}^{-1}$ at 50.2° (3 s.f.) to the line of centres
$\frac{1}{6}v\,\text{m s}^{-1}$ along the line of centres
6 a


Perpendicular to the line of centres, component of velocity of A is $u\sin\alpha$. Parallel to the line of centres:
conservation of momentum: $mu\cos\alpha = mv + mw$,
$u\cos\alpha = v + w$
law of restitution: $w - v = eu\cos\alpha$,
so $2v = u\cos\alpha - eu\cos\alpha = u\cos\alpha(1-e)$

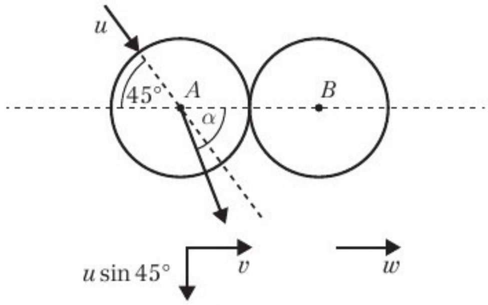
$$\Rightarrow \tan\beta = \frac{u\sin\alpha}{v} = \frac{2u\sin\alpha}{u\cos\alpha(1-e)} = \frac{2\tan\alpha}{1-e}$$

b The path of A has been deflected through an angle equal to $\beta - \alpha$.

$$\tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta} = \frac{\frac{2\tan\alpha}{1-e} - \tan\alpha}{1 + \tan\alpha\frac{2\tan\alpha}{1-e}}$$

$$= \frac{2\tan\alpha - (1-e)\tan\alpha}{1 - e + 2\tan^2\alpha}$$

$$\beta - \alpha = \arctan\left(\frac{(1+e)\tan\alpha}{2\tan^2\alpha + 1 - e}\right)$$

7 a 2.53 J (3 s.f.) b 3.06 Ns (3 s.f.)
8 a $\sqrt{13}\,\text{m s}^{-1}$, $2\sqrt{5}\,\text{m s}^{-1}$ b $\frac{10}{43}$
9 $4\mathbf{i} + \mathbf{j}\,\text{m s}^{-1}$, $2\mathbf{i}\,\text{m s}^{-1}$
10 $3.23\,\text{m s}^{-1}$, $3.25\,\text{m s}^{-1}$ (3 s.f.)
11 1.92 J (3 s.f.)
12 a $\left(\frac{5}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right)\text{m s}^{-1}$ b $\frac{1}{\sqrt{10}}(\mathbf{i} - 3\mathbf{j})$
13 a $\sqrt{5}\,\text{m s}^{-1}$ b $9m$ J
14 a 0 b $\frac{1}{7}$
15 $1\,\text{m s}^{-1}$, $\sqrt{13}\,\text{m s}^{-1}$
16


Parallel to the line of centres, using conservation of momentum and the law of restitution gives
$mu\cos 45° = mv + mw$ and $w - v = eu\cos 45°$
By subtracting

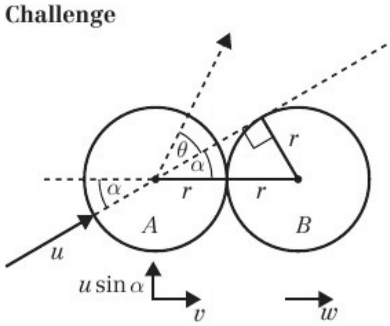
$$2v = u\cos 45°(1-e)$$

$$v = \frac{u\sqrt{2}(1-e)}{4}$$

$$\Rightarrow \tan\alpha = \frac{u\sin 45°}{\left(\frac{u\sqrt{2}(1-e)}{4}\right)} = \frac{2}{1-e}$$

$$\theta = \alpha - 45° \Rightarrow \tan\theta = \frac{\tan\alpha - \tan 45°}{1 + \tan\alpha\tan 45°} = \frac{\frac{2}{1-e} - 1}{1 + \frac{2}{1-e}}$$

$$= \frac{2 - 1 + e}{1 - e + 2} = \frac{1+e}{3-e}$$

Challenge


Tangent perpendicular to radius $\Rightarrow \sin\alpha = \frac{1}{2}$

Initial components of velocity of A are $u\cos\alpha$ parallel to the line of centres, and $u\sin\alpha$ perpendicular to the line of centres.

momentum $\Rightarrow mu\cos\alpha = mv + mw$

$u\cos\alpha = v + w$

impact $\Rightarrow w - v = eu\cos\alpha$

Subtracting gives

$2v = u\cos\alpha - eu\cos\alpha$

$$v = \frac{u\cos\alpha\left(1-\frac{1}{2}\right)}{2} = \frac{u \times \frac{\sqrt{3}}{2} \times \frac{1}{2}}{2} = \frac{u\sqrt{3}}{8}$$

$$\Rightarrow \tan(\theta+\alpha) = \frac{u\sin\alpha}{v} = \frac{\left(\frac{u}{2}\right)}{\left(\frac{u\sqrt{3}}{8}\right)} = \frac{4}{\sqrt{3}}$$

$$\tan\theta = \frac{\tan(\theta+\alpha) - \tan\alpha}{1 + \tan(\theta+\alpha)\tan\alpha} = \frac{\frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \frac{4}{\sqrt{3}} \times \frac{1}{\sqrt{3}}} = \frac{\left(\frac{3}{\sqrt{3}}\right)}{\left(\frac{3+4}{3}\right)}$$

$$= \frac{3\sqrt{3}}{7}$$

Mixed exercise 5

1 $\frac{\sqrt{7}}{5}$

2 a $2.33\,\text{m s}^{-1}$ b $3\,\text{N s}$

3 a $\frac{1}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}$ b $3.375\,\text{J}$ c $128°$ (3 s.f.)

4 $\frac{2}{39}$

5 a $1.80\,\text{m s}^{-1}$ (3 s.f.) at an angle of $16.1°$ (3 s.f.) to the first wall.

b $1\,\text{m s}^{-1}$ parallel to the original path but in the opposite direction.

c The resistance to motion will reduce the final speed of the sphere but the angle of motion will remain the same.

6 a 0.721

b $18.0\,\text{m s}^{-1}$ parallel to the original path but in the opposite direction.

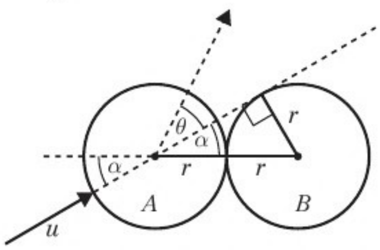
7 a $0.386\,\text{J}$ (3 s.f.)

b $1.28\,\text{m s}^{-1}$ (3 s.f.) at an angle of $55.6°$ (3 s.f.) to the second wall.

8 a $-\mathbf{i} + 3\mathbf{j}$

b $\frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$

9



Tangent perpendicular to radius $\Rightarrow \sin\alpha = \frac{1}{2}$

Inititial components of velocity of A are $u\cos\alpha$ parallel to the line of centres, and $u\sin\alpha$ perpendicular to the line of centres.

Momentum $\Rightarrow mu\cos\alpha = mv + mw$

$u\cos\alpha = v + w$

Impact $\Rightarrow w - v = \text{eu}\cos\alpha$

where v is the velocity of a along the line of centres and w the velocity of B along the line of centres immediately after the collision.

Subtracting gives $2v = u\cos\alpha - eu\cos\alpha$,

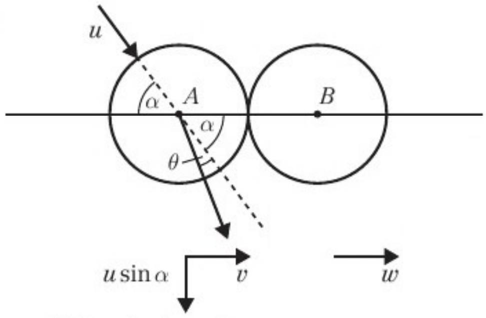
$$v = \frac{u\cos\alpha\left(1-\frac{2}{3}\right)}{2} = \frac{u \times \frac{\sqrt{3}}{2} \times \frac{1}{3}}{2} = \frac{u\sqrt{3}}{12}$$

$$\Rightarrow \tan(\theta+\alpha) = \frac{u\sin\alpha}{v} = \frac{\left(\frac{u}{2}\right)}{\left(\frac{u\sqrt{3}}{12}\right)} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

$$\tan\theta = \frac{\tan(\theta+\alpha) - \tan\alpha}{1 + \tan(\theta+\alpha)\tan\alpha} = \frac{2\sqrt{3} - \frac{1}{\sqrt{3}}}{1 + 2\sqrt{3} \times \frac{1}{\sqrt{3}}} = \frac{\left(\frac{6-1}{\sqrt{3}}\right)}{(1+2)}$$

$$= \frac{5}{3\sqrt{3}} + \frac{5\sqrt{3}}{9}$$

10



Parallel to the line of centres, using conservation of momentum and the impact law gives

$mu\cos\alpha = mv + mw$ and $w - v = eu\cos\alpha$

By subtracting

$2v = u\cos\alpha \times (1 - e)$

$$v = \frac{4u(1-e)}{10} = \frac{2u(1-e)}{5}$$

$$\Rightarrow \tan(\theta+\alpha) = \frac{u\sin\alpha}{\left(\frac{2u(1-e)}{5}\right)}$$

$$= \frac{3}{2(1-e)}$$

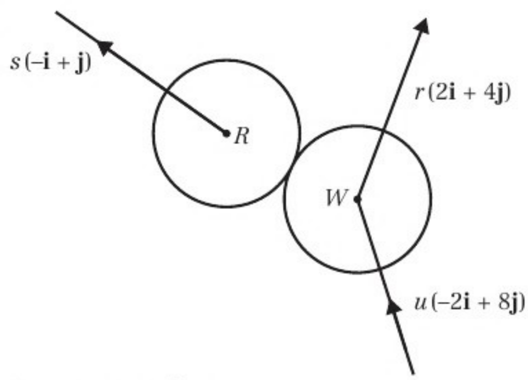
$$\tan\theta = \frac{\tan(\theta+\alpha) - \tan\alpha}{1 + \tan(\theta+\alpha)\tan\alpha} = \frac{\frac{3}{2(1-e)} - \frac{3}{4}}{1 + \frac{3}{2(1-e)} \times \frac{3}{4}}$$

$$= \frac{12 - 6(1-e)}{8(1-e) + 9}$$

$$= \frac{6 + 6e}{17 - 8e}$$

11 $\sqrt{5}\,\text{m s}^{-1}$, $1.5\,\text{m s}^{-1}$

12



Conservation of momentum:

$u(-2\mathbf{i} + 8\mathbf{j}) = s(-\mathbf{i} + \mathbf{j}) + r(2\mathbf{i} + 4\mathbf{j})$

$\Rightarrow -2u = -s + 2r$ and $8u = s + 4r$

Adding $\Rightarrow 6u = 6r, r = u, s = 4u$
Line of centres is parallel to $-\mathbf{i} + \mathbf{j}$ (as this is the direction of the impulse on the red ball).
In the direction of the line of centres
component of $(-2\mathbf{i} + 8\mathbf{j})$ is $\frac{(-2\mathbf{i} + 8\mathbf{j})(-\mathbf{i} + \mathbf{j})}{|(-\mathbf{i} + \mathbf{j})|}$
$= \frac{10\sqrt{2}}{2} = 5\sqrt{2}$ component of $(2\mathbf{i} + 4\mathbf{j})$ is $\frac{2\sqrt{2}}{2} = \sqrt{2}$ and
component of $(-\mathbf{i} + \mathbf{j})$ is $\sqrt{2}$
so using law of restitution:
$4u\sqrt{2} - u\sqrt{2} = e \times 5u\sqrt{2} \Rightarrow 3\sqrt{2} = 5\sqrt{2}e$
$\Rightarrow e = \frac{3}{5}$

13 $\frac{3u}{2}, \frac{3\sqrt{29}}{10}u$
Directions relative to the line of centres are
$\arctan\left(\frac{\frac{6}{5}}{\frac{9}{10}}\right) = \arctan\frac{4}{3}$ and $\arctan\left(\frac{\frac{3}{5}}{\frac{3}{2}}\right) = \arctan\frac{2}{5}$, so the angle between the paths is:
$\arctan\left(\frac{\frac{4}{3} - \frac{2}{5}}{1 + \frac{4}{3} \times \frac{2}{5}}\right) = \arctan\left(\frac{20 - 6}{15 + 8}\right) = \arctan\frac{14}{23}$

Challenge

a Collision 1: $\tan a = e \tan\theta \Rightarrow a = 35.54°$
Collision 2: angle of approach $= 45° + a = 80.54°$
: $\tan b = 0.5 \tan(80.54°) = 3 \Rightarrow b = 71.56°$
Collision 3: angle of approach $= 180° - 45° - 71.56°$
$= 63.43°$
: $\tan c = 0.5 \tan(63.43°) = 1 \Rightarrow c = 45°$
Angle of deflection parallel to wall, so no fourth collision.

b 94.6% of K.E. lost (3 s.f.)

Review exercise 2

1 $e = \frac{3}{5}$

2 **a** Using conservation of momentum:
$mu + km\lambda u = kmv$
$u(1 + k\lambda) = kv$ (1)
Newton's law of restitution:
$v = e(u - \lambda u) = eu(1 - \lambda)$ (2)
Eliminate v from (1) and (2)
$u(1 + k\lambda) = keu(1 - \lambda)$
$e = \frac{1 + k\lambda}{k(1 - \lambda)}$

b $e \leqslant 1$
$\Rightarrow 1 + k\lambda \leqslant k - k\lambda$
$\frac{1}{1 - 2\lambda} \leqslant k$
but $0 < \lambda < \frac{1}{2} \Rightarrow 0 < 1 - 2\lambda < 1$ and $k > 1$

3 **a** Using conservation of momentum:
$mu = mv_S + 2mv_T$
$u = v_S + 2v_T$ (1)
Newton's law of restitution:
$eu = v_T - v_S$ (2)
Eliminating v_S from (1) and (2):
$u + eu = 3v_T$
$v_T = \frac{1}{3}u(1 + e)$

b **i** Speed of S is $\frac{1}{3}u(2e - 1)$
ii Direction of motion reversed.

4 **a** Using conservation of momentum:
$3m \times 2u - 2mu = 3mu_P + 2mu_Q$
$\therefore 4u = 3u_P + 2u_Q$ (1)
Newton's law of restitution:
$e(2u + u) = u_Q - u_P$
$3eu = u_Q - u_P$ (2)
Eliminate u_P from (1) and (2)
$4u = 3(u_Q - 3eu) + 2u_Q$
$4u = 5u_Q - 9eu$
$u_Q = \frac{1}{5}u(9e + 4)$

b $\frac{2}{3} < e \leqslant 1$ **c** $e = \frac{7}{9}$

5 14.4 m

6 **a** $e = \frac{2}{3}$
b If the plane was rough and the conditions described above are otherwise identical, then the coefficient of restitution must be greater. If it were the same then the speed of the particle at each point on the return journey would be $< \frac{2}{3}$ of the minimum speed of the particle on the outward journey.

7 **a** $\frac{\sqrt{70}}{10}$
b Time t_1 until first bounce
$50 = \frac{9.8}{2}t_1^2 \Rightarrow t_1 = \frac{10\sqrt{5}}{7}$
Time t_2 to reach maximum height after first bounce
$35 = \frac{9.8}{2}t_2^2 \Rightarrow t_2 = \frac{5\sqrt{14}}{7}$
Total time from first to second bounce
$T = t_1 + 2t_2$
$= \frac{10\sqrt{5}}{7} + 2 \times \frac{5\sqrt{14}}{7} = \frac{10}{7}(\sqrt{5} + \sqrt{14})$ s

8 **a** A has speed $\frac{u}{2}$
B has speed $\frac{3u}{2}$
b $\frac{45}{2}mu^2$ J

9 **a** $e = \frac{1}{4}$ **b** $\frac{9}{4}mu^2$ J

10 **a** Using conservation of momentum:
$mu = mv_A + 3mv_B$
$u = v_A + 3v_B$ (1)
Newton's law of restitution:
$eu = v_B - v_A$ (2)
Eliminating v_A from (1) and (2) gives $u + eu = 4v_B$
$v_B = \frac{1}{4}(1 + e)u$
b $v_A = \frac{1}{4}(1 - 3e)u$
c $e = \frac{1}{3}$
d $v_A = \frac{u}{4}(1 - 3e) = \frac{u}{4}\left(1 - 3 \times \frac{1}{3}\right) = 0$
$\therefore A$ is at rest

11 41.5 kJ

12 $\frac{1}{12}$ J

13 **a** $e = \frac{2}{3}$
b From part **a**, $v_A = -\frac{2}{3}u$
For spheres B and C:
Conservation of momentum:
$4mu = 2mv_B + 5mv_C$
Newton's law of restitution:
$\frac{v_C - v_B}{2u} = \frac{3}{5}$
$v_C - v_B = \frac{6}{5}u$
$v_C = v_B + \frac{6}{5}u$
Substitute v_C in the conservation of momentum equation:
$4u = 2v_B + 6u + 5v_B$
$\Rightarrow v_B = -\frac{2}{7}u$
$\Rightarrow v_C = \frac{32}{35}u$
$v_A < v_B < v_C$ $\therefore$ there are no further collisions.

14 **a** Using conservation of momentum:
$2mu = 2mv_P + mv_Q$
$2u = 2v_P + v_Q$ (1)
Newton's law of restitution:
$\frac{1}{3}u = v_Q - v_P$ (2)

$(1) + 2 \times (2): \frac{8u}{3} = v_Q + 2v_Q$

$3v_Q = \frac{8u}{3}$

$v_Q = \frac{8u}{9}$

Using (2)

$v_P = v_Q - \frac{1}{3}u$

$v_P = \frac{8u}{9} - \frac{1}{3}u$

$v_P = 5\frac{u}{9}$

b $e = \frac{25}{32}$

c Q is now moving towards the wall; after it rebounds off the wall it will return to collide with P once more.

15 a Using conservation of momentum:
$2m \times 5u = 2mv_P + 3mv_Q$
$10u = 2v_P + 3v_Q$ (1)
Newton's law of restitution:
$e \times 5u = v_Q - v_P$ (2)
(1) + 2 × (2)
$10u + 10eu = 3v_Q + 2v_Q$
$10u + 10eu = 5v_Q$
$v_Q = 2u + 2eu = 2(1 + e)u$

b From (2)
$v_P = v_Q - 5eu = 2(1 + e)u - 5eu$
$v_P = 2 \times 1.4u - 5 \times 0.4u = 0.8u$
$v_P > 0 \therefore P$ moves towards wall and will collide with Q after Q rebounds from the wall.

c $e = 0.8$
$v_P = 2 \times 1.8u - 5 \times 0.8u = -0.4u$
Q hits the wall
$v_Q = 3.6uf$
For a second collision
$3.6uf > 0.4u$
$f > \frac{0.4}{3.6} = \frac{1}{9}$
Range of values for f is
$\frac{1}{9} < f \leqslant 1$

16 a Using conservation of momentum:
$2m \times 2u + 3m \times u = 2mv_A + 3mv_B$
$7u = 2v_A + 3v_B$ (1)
Newton's law of restitution:
$e(2u - u) = v_B - v_A$
$eu = v_B - v_A$ (2)
(1) + 2 × (2)
$7u + 2eu = 3v_B + 2v_B$
$v_B = \frac{1}{5}u(7 + 2e)$

b $v_A = \frac{1}{5}u(7 - 3e)$

c $\frac{1}{5}u(7 - 3e) = \frac{11u}{10}$
$14u - 6eu = 11u \Rightarrow 6eu = 3u$
$e = \frac{1}{2}$

d $\frac{5d}{16}$

e After B hits the barrier
$v_B = \frac{11}{16} \times \frac{8u}{5} = \frac{11u}{10}$
Equal speeds, opposite directions.
$\therefore A$ and B will collide at midpoint of the distance from A to the barrier at the instant B hits the barrier, i.e. they collide at distance $\frac{5d}{32}$ from the barrier.

17 a $\frac{82}{9}$ m

b This model predicts an infinite number of bounces which is not realistic.

18 First collision: A with B
Using conservation of momentum:
$4m = 2mv_B + mv_A$
$4 = 2v_B + v_A$ (1)
Newton's law of restitution:
$4 \times 0.7 = 2.8 = v_B - v_A$ (2)
(1) + (2)
$6.8 = 3v_B$
$v_B = \frac{34}{15}\text{ m s}^{-1}$
$v_A = -\frac{8}{15}\text{ m s}^{-1}$
Second collision: B with C
$\frac{68}{15} = 2v'_B + 3v_C$ (3)
$\frac{34}{15} \times 0.4 = \frac{68}{75} = v_C - v'_B$ (4)
(3) + 2 × (4)
$\frac{476}{75} = 5v_C$
$v_C = \frac{476}{375}\text{ m s}^{-1}$
$v'_B = \frac{136}{375}\text{ m s}^{-1}$
$v_A < v'_B < v_C \therefore$ there are no further collisions.

19 Kinetic energy $= \frac{1}{2}mu^2(e^2 \sin^2\alpha + \cos^2\alpha)$

20 a $\frac{2}{3}$ **b** $\frac{1}{3}$

21 a 60°

b $\tan\beta = e\tan\alpha$
$\Rightarrow e = \frac{\tan\beta}{\tan\alpha} = \frac{\tan 30°}{\tan 60°}$
$\sqrt{3}$
$\Rightarrow e = \frac{3}{\sqrt{3}} = \frac{1}{3}$

22 0.657 (3 s.f.)

23 a $\frac{1}{2}\sqrt{10}$ N s in the direction parallel to the unit vector $\frac{1}{\sqrt{10}}(-\mathbf{i} - 3\mathbf{j})$

b 2 J

24 a $e = \frac{3}{4}$ **b** 11 m s^{-1}

25 a Angle of of 8.1° (2 s.f.) to first wall

b $e = \frac{1}{7}$ **c** 1.3×10^{-3} J

26 0.41 J

27 a $\frac{\sqrt{15}}{6}$

b 2.34 m s^{-1} (3 s.f.) at an angle of 17.4° (3 s.f.) to W_2.

28 19.5m J

29 a 1.43 m s^{-1}
Moving away from the first wall at an angle of 24.8°

b $e = 0.81$

30 26.2 J

31 a Perpendicular to the line of centres CB:
In this direction the component of the velocity of P is unchanged and is
$\frac{13}{12}u\sin\alpha = \frac{13}{12}u \times \frac{5}{13} = \frac{5}{12}u$
Parallel to the line of centres CB:
Conservation of linear momentum:
$m \times \frac{13}{12}u\cos\alpha = -mx + 2m \times \frac{3}{5}u$
$x = \frac{6}{5}u - \frac{13}{12}u \times \frac{12}{13} = \frac{1}{5}u$

b $e = \frac{4}{5}$

c Let the time after the collision for Q to reach C be t_1
distance = speed × time
$d_1 = \frac{3}{5}ut_1 \Rightarrow t_1 = \frac{5d_1}{3u}$
Perpendicular to W, in time t_1, P travels a distance s given by
distance = speed × time
$s = \frac{1}{5}u \times t_1 = \frac{1}{5}u \times \frac{5d_1}{3u} = \frac{1}{3}d_1$
The distance from P from W is
$d_1 + s = d_1 + \frac{1}{3}d_1 = \frac{4}{3}d_1$

d Before hitting W, Q has speed $\frac{3}{5}u$.
After hitting W, Q has speed $e \times \frac{3}{5}u = \frac{1}{2} \times \frac{3}{5}u = \frac{3}{10}u$
In the direction CB, the velocity of Q relative to P is
$\frac{3}{10}u - \frac{1}{5}u = \frac{1}{10}u$
The time, t_2, for Q to travel from C to the point of the second collision is given by
$$t_2 = \frac{\frac{4}{3}d_1}{\frac{1}{10}u} = \frac{40d_1}{3u}$$
The time between the two collisions is
$$t_1 + t_2 = \frac{5d_1}{3u} + \frac{40d_1}{3u} = \frac{15d_1}{u}$$
e 12 : 25

32 $\left(-\frac{23}{9}\mathbf{i} - \frac{46}{9}\mathbf{j}\right)\,\mathrm{m\,s^{-1}}$

33 $e = \frac{1}{3}$

34 Let the mass of each sphere be m.
Let the speed of Q immediately before the collision be u and its speed immediately after the collision be v.
Let the speed of P immediately after the collision be x
Perpendicular to the line of centres:
For Q
$u \sin\alpha = v \sin\beta$ (1)
Along the line of centres:
Conservation of linear momentum:
$mu \cos\alpha = mv \cos\beta + mx$
$x = u \cos\alpha - v \cos\beta$ (2)
Newton's law of restitution:
$x - v \cos\beta = eu \cos\alpha$
$x = eu \cos\alpha + v \cos\beta$ (3)
Eliminating x between (2) and (3)
$u \cos\alpha - v \cos\beta = eu \cos\alpha + v \cos\beta$
$(1 - e)u \cos\alpha = 2v \cos\beta$ (4)
Dividing (1) by (4)
$$\frac{u \sin\alpha}{(1-e)u \cos\alpha} = \frac{v \sin\beta}{2v \cos\beta}$$
$$\frac{\tan\alpha}{1-e} = \frac{\tan\beta}{2}$$
$(1 - e)\tan\beta = 2 \tan\alpha$

35 **a** Perpendicular component of velocity of A immediately before the collision is $2\,\mathrm{m\,s^{-1}}$.
Parallel component of velocity of A immediately before the collision is $1.5\,\mathrm{m\,s^{-1}}$.
Perpendicular component of velocity of B immediately before the collision is $1.2\,\mathrm{m\,s^{-1}}$.
Parallel component of velocity of B immediately before the collision is $0.5\,\mathrm{m\,s^{-1}}$.
b The speed of A is $2.1\,\mathrm{m\,s^{-1}}$.
The speed of B is $1.9\,\mathrm{m\,s^{-1}}$.

36 **a** Let the mass of each sphere be m.
Let the components of the velocity of T be x and $U\cos\alpha$.
Let the velocity of S be y.
Perpendicular to the line of centres:
The component of the velocity is unchanged, so the component of the velocity of T after the impact perpendicular to the line of centres is $U\cos\alpha$.
Parallel to the line of centres:
Conservation of linear momentum:
$mU \sin\alpha = mx + my$
$x + y = U \sin\alpha$ (1)
Newton's law of restitution:
$y - x = eU \sin\alpha$ (2)
(1) – (2):
$2x = U \sin\alpha - eU \sin\alpha = U(1 - e)\sin\alpha$
$x = \frac{1}{2}U(1 - e)\sin\alpha$

b Let the components of the velocity of T after the impact, parallel and perpendicular to the wall, be X and Y respectively.
R(↓): $X = U\cos\alpha \sin\alpha - x\cos\alpha$
$= U\cos\alpha \sin\alpha - \frac{1}{2}U(1-e)\sin\alpha \cos\alpha$
$= U\cos\alpha \sin\alpha\left(1 - \frac{1}{2} + \frac{1}{2}e\right) = U\cos\alpha \sin\alpha\left(\frac{1}{2} + \frac{1}{2}e\right)$
$= \frac{1}{2}U(1+e)\cos\alpha \sin\alpha$
R(→): $Y = U\cos\alpha \cos\alpha + x\sin\alpha$
$= U\cos^2\alpha + \frac{1}{2}U(1-e)\sin\alpha \sin\alpha$
$= U(1 - \sin^2\alpha) + \frac{1}{2}U(1-e)\sin^2\alpha$
$= \frac{1}{2}U(2 - 2\sin^2\alpha + \sin^2\alpha - e\sin^2\alpha)$
$= \frac{1}{2}U(2 - \sin^2\alpha - e\sin^2\alpha)$
$= \frac{1}{2}U(2 - (1+e)\sin^2\alpha)$

c $\frac{40}{21}d$

37 **a** A $(3\mathbf{i} + \mathbf{j})\,\mathrm{m\,s^{-1}}$ B $(2\mathbf{i} - 3\mathbf{j})\,\mathrm{m\,s^{-1}}$
b $4m\,\mathrm{N\,s}$ **c** 37°

38 **a** $\sqrt{2}mu$ **b** d

39 **a** Let the components of velocities parallel to the line of centres for A and B be x and y respectively.
Parallel to the line of centres:
Conservation of linear momentum:
$mu\cos60° = mx + kmy$
$x + ky = \frac{1}{2}u$ (1)
Newton's law of restitution:
$y - x = \frac{1}{2}u\cos60° = \frac{1}{4}u$ (2)
$$(1) + (2): ky + y = \frac{3u}{4} \Rightarrow y = \frac{3u}{4k+1}$$
b From (2) in part **a**,
$$x = y - \frac{u}{4} = \frac{3u}{4k+1} - \frac{u}{4} = \frac{3u - u(k+1)}{4(k+1)} = \frac{(2-k)u}{4(k+1)}$$
The direction of motion of A is given by
$$\tan\theta = \frac{u\sin60°}{x} = \frac{\sqrt{3}}{2}u \times \frac{4(k+1)}{(2-k)u}$$
Given that $\tan\theta = 2\sqrt{3}$
$$2\sqrt{3} = \frac{4(k+1)\sqrt{3}}{2(2-k)}$$
$k + 1 = 2 - k$
$2k = 1 \Rightarrow k = \frac{1}{2}$
c $\frac{1}{32}mu^2$

Challenge

1 At the time of collision with B, A is
$6u \times \frac{l}{11u} = \frac{6l}{11}$ m from W_1.
After time T_1 the balls collide:
$6uT_1 + 5uT_1 = l$
Collision A with B:
Conservation of momentum:
$6mu - 5mu = mv_B - mv_A$
$u = v_B - v_A$ (1)
Newton's law of restitution:
$v_B + v_A = 11eu$ (2)
$v_B = 11eu - v_A$
So $u = 11eu - 2v_A$
$$v_A = \frac{11eu - u}{2} = \frac{u}{2}(11e - 1)$$
Time from collision with B to W_1
$$T_2 = \frac{\frac{6l}{11}}{\frac{u}{2}(11e-1)} = \frac{12l}{11u(11e-1)}$$
$$T = T_1 + T_2 = \frac{l}{11u} + \frac{12l}{11u(11e-1)}$$
$$= \frac{l(11e+11)}{11u(11e-1)} = \frac{l(e+1)}{u(11e-1)}$$

2 **a** **i** 2.25 seconds
ii 3.52 seconds (3 s.f.)
b 30 100 seconds, 0.000 166 m s^{-1} (3 s.f.)
c In reality the surface would not be smooth and the coefficient of restitution would be 0 for sufficiently small values of v, so the ball would have stopped moving by this time.

3 **a** 9 m
b Initial K.E. $\frac{1}{2}m \times 100$ J
Taking component after rebound parallel and perpendicular to the plane, after first bounce velocity is $\sqrt{6^2 + 4^2}$ so K.E. is $\frac{1}{2}m \times 52$ J.
Percentage loss in K.E. at first bounce is $\frac{100 - 52}{100} = 48\%$
From part **a**, ball rebounds after first bounce with $v_H = \frac{36}{5}$ m s^{-1} and $v_V = \frac{2}{5}$ m s^{-1} (downwards).
After 1 second, when ball meets the plane a second time, the velocity is
$v_H = \frac{36}{5}$ m s^{-1} and $v_V = \frac{2}{5} + 10 = \frac{52}{5}$ m s^{-1}
So K.E. before second bounce is
$\frac{1}{2}m\left(\frac{36^2}{25} + \frac{52^2}{25}\right) = 80m$ J
After second bounce:
Velocity parallel to plane:
$v_\parallel = \frac{52}{5}\sin\theta + \frac{36}{5}\cos\theta = 12$ m s^{-1}
Velocity perpendicular to plane:
$v_\perp = \frac{1}{2}\left(\frac{36}{5}\sin\theta - \frac{52}{5}\cos\theta\right) = 2$ m s^{-1}
So K.E. after second bounce $= \frac{1}{2}m(12^2 + 2^2) = 74m$ J
Percentage loss in K.E. at second bounce is $\frac{6}{80} = 7.5\%$
Angle of incidence with plane is much shallower on second bounce than on first bounce.

Exam-style practice: AS level

1 **a** **i** 169 J
ii $R = 14.1$ N (3 s.f.)
b 8.51 m s^{-1} (3 s.f.)

2 7.35 m s^{-1} (3 s.f.)

3 **a** A has speed 0.8 m s^{-1},
B has speed 2.4 m s^{-1}. Both balls move in the direction A was originally travelling in.
b 0.75 kg

4 **a** $\lambda = mv - mu$ and $v = eu$
$\lambda = meu - mu = mu(1 + e)$
b Let time to collision with W_2 be T_1:
$T_1 = \frac{l}{u}$
Let time to collision with W_1 be T_2:
$T_2 = \frac{l}{eu}$
Total time:
$T_1 + T_2 = \frac{l}{u} + \frac{l}{eu} = \frac{el + l}{eu} = \frac{l}{eu}(e + 1)$

5 Collision 1: A with B
$3m = 2mv_B + mv_A$
$v_B - v_A = 2.7$
$3 = 5.4 + 3v_A$
$v_A = -0.8$ m s^{-1} and $v_B = 1.9$ m s^{-1}
Collision 2: B with C
$3.8m = 3mv_C + 2mv_B$
$v_C - v_B = 1.71$
$3.8 = 5.13 + 5v_B$
$v_B = -0.266$ m s^{-1} and $v_C = 1.444$ m s^{-1}
$v_A < v_B < v_C$
Therefore there are no further collisions.

Exam-style practice: A level

1 **a** 0 **b** 1.5 m s^{-1}
c 7.5 J **d** 0.057 (2 s.f.)

2 **a** 0.44 m s^{-2}
b Tractive force:
$T = 20\,000 \div 20 = 1000$ N
Total driving forces:
$1000 + 1400g \sin 6° = 2434$ N
Non-gravitational resistances:
$120 + 2(20^2) = 920$ N
Total driving forces larger than resistive forces, driver will have to brake (add another resistive force) to maintain speed.
c 25.6 m s^{-1}

3 $e = 0.468$ (3 s.f.)

4 **a** $\frac{3\sqrt{5}}{5}$ **b** 63.4° (3 s.f.)

5 **a** 1.4 m (2 s.f.) **b** 2.8 J (2 s.f.)
c 3.6 m s^{-2} (2 s.f.)
d h = distance above lowest point
P.E. gained = E.P.E lost
$0.25 \times 9.8 \times h = \frac{15 \times 0.7^2}{2.4} \Rightarrow h = 1.25$
The particle reaches a height of 1.25 m above the starting point. This is 0.65 m above the ceiling.

6 **a** **i** P: 2.1 m s^{-1} in the direction it was originally moving
Q: 2.7 m s^{-1} in the direction P was originally moving
ii 0.2
b 2.33 J (3 s.f.)
c P and Q are moving in the same direction. P is moving at 2.1 m s^{-1} and Q is moving at 0.54 m s^{-1}. So yes, they will collide again.

7 **a** **i** 57.8° (3 s.f.)
ii 0.916 (3 s.f.)
b S has velocity $0.9795u$ m s^{-1} at 85.5° to the line of centres and T has velocity $0.4375u$ m s^{-1} along the line of centres.

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